Probability and Statistics

PhD Qualification Exam

Mock Examination No. 4

Total: 100 points

- There are eight compulsory problems. The point values sum to 100.
- Give all necessary details. Unsupported final answers receive little credit.
- You may quote standard theorems only after verifying their assumptions.

Problem 1 **10 points**

Let A and B be independent and uniform on the finite field $\mathbb{F}_3 = \{0, 1, 2\}$, with arithmetic modulo 3. Define

$$X = A, \qquad Y = B, \qquad Z = A + B \pmod 3.$$

(a) Prove that every pair among X, Y, Z is independent. [5]

(b) Prove that X, Y, Z are not mutually independent. [3]

(c) If the values are viewed as real numbers $0, 1, 2$, compute all pairwise covariances and explain why the result does not settle mutual independence. [2]

Problem 2 **15 points**

(a) Let $X_n \xrightarrow{p} X$ and suppose the family $\{X_n : n \geq 1\}$ is uniformly integrable. Prove that X is integrable and $X_n \to X$ in L^1. State clearly where uniform integrability is used. [10]

(b) Let $U \sim \text{Uniform}(0, 1)$ and $X_n = n\mathbf{1}_{\{U \leq 1/n\}}$. Show that $X_n \to 0$ almost surely but not in L^1, and verify directly that $\{X_n\}$ is not uniformly integrable. [5]

Problem 3 **10 points**

Let $S_n = \sum_{k=1}^{n} \xi_k$, where the ξ_k are independent with $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = 1/2$, and let

$$\tau = \inf\{n \geq 0 : |S_n| = a\}, \qquad a \in \mathbb{N}.$$

(a) Prove that $\tau < \infty$ almost surely and compute $\mathbb{E}\tau$ using the martingale $S_n^2 - n$. Verify the optional-stopping assumptions rather than quoting the theorem without comment. [7]

(b) Compute $\mathbb{P}(S_\tau = a)$. [3]

Problem 4 **15 points**

Let Q be an arbitrary transition matrix on a finite state space S. Let ν be a probability distribution satisfying $\nu_j > 0$ for every $j \in S$, and fix $0 < \alpha < 1$. Define

$$P(i, j) = (1 - \alpha)Q(i, j) + \alpha\nu_j.$$

(a) Prove that P is irreducible and aperiodic. [3]

(b) Prove that

$$\pi = \alpha \sum_{k=0}^{\infty} (1 - \alpha)^k \nu Q^k$$

is a stationary distribution. [5]

(c) Prove uniqueness and the quantitative bound

$$\|\mu P^n - \pi\|_{\mathrm{TV}} \le (1 - \alpha)^n \|\mu - \pi\|_{\mathrm{TV}}$$

for every initial distribution μ. [7]

Problem 5 **15 points**

For $\theta \in \mathbb{R}$, consider the density

$$f_\theta(x) = \frac{e^{\theta x}}{Z(\theta)} \mathbf{1}_{[-1,1]}(x), \qquad Z(\theta) = \int_{-1}^{1} e^{\theta u}\, du.$$

Let $X_1, \ldots, X_n$ be i.i.d., and let $0 < \alpha < 1/2$.

(a) Compute $Z(\theta)$, including its value at $\theta = 0$, and prove that the family has a monotone likelihood ratio in $T = \sum_i X_i$. [4]

(b) Construct a size-α UMP test of $H_0 : \theta \le 0$ versus $H_1 : \theta > 0$. Express the critical value using the distribution of T under $\theta = 0$ and justify the size over the composite null. [6]

(c) Prove that no size-α UMP test exists for $H_0 : \theta = 0$ versus $H_1 : \theta \ne 0$. [5]

Problem 6 **10 points**

Let $X_1, \ldots, X_n$ be i.i.d. Uniform$(0, \theta)$, where $\theta > 0$, and let $M = X_{(n)}$.

(a) Show that $\delta = (n + 1)M/n$ is unbiased for θ and compute its variance. [5]

(b) A mechanical calculation using the interior score $-1/\theta$ appears to give Fisher information n/θ^2 and hence the lower bound θ^2/n. Explain rigorously why this Cramér–Rao argument is invalid, and identify the failed regularity condition. [5]

Problem 7 **15 points**

Let $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathbb{R}^n$ be fixed linear subspaces of dimensions $r_0 < r_1 < n$, with orthogonal projections P_0, P_1. Suppose

$$Y \sim N_n(\mu, \sigma^2 I_n), \qquad \mu \in \mathcal{M}_1,$$

where σ^2 is unknown. Define $\mathrm{RSS}_j = \|(I - P_j)Y\|^2$.

(a) Under $H_0 : \mu \in \mathcal{M}_0$, derive the joint distribution and independence of $\mathrm{RSS}_0 - \mathrm{RSS}_1$ and RSS_1. [6]

(b) Construct the exact level-α F test of H_0 against $\mu \in \mathcal{M}_1 \setminus \mathcal{M}_0$. [4]

(c) Under a fixed $\mu \in \mathcal{M}_1$, identify the noncentrality parameter and the distribution governing the power. [5]

Problem 8 **10 points**

Let $X \mid p \sim \mathrm{Binomial}(n, p)$ and put the uniform prior on $p \in (0, 1)$. The loss is absolute error $L(p, d) = |p - d|$.

(a) Find the posterior distribution and characterize the Bayes estimator. [5]

(b) If the observation is $X = n$, compute the Bayes estimator explicitly. [3]

(c) Compare it with the posterior mean and explain why the two need not agree. [2]