Probability and Statistics

PhD Qualification Exam

Mock Examination No. 3

Total: 100 points

- There are eight compulsory problems. The point values sum to 100.
- Give all necessary details. Unsupported final answers receive little credit.
- You may quote standard theorems only after verifying their assumptions.

Problem 1 — 10 points

Let $Z \sim N(0,1)$ and let ε be independent of Z with $\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = 1/2$. Define $X = Z$ and $Y = \varepsilon Z$.

(a) Find the marginal distributions of X and Y, and compute $\mathrm{Cov}(X, Y)$. [3]

(b) Prove that X and Y are not independent and that (X, Y) is not jointly Gaussian. [3]

(c) Compute $\mathbb{E}(X \mid Y)$ and $\mathbb{E}(X^2 \mid Y)$. [4]

Problem 2 — 10 points

Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}X_1 = 0$ and $0 < \mathrm{Var}(X_1) = \sigma^2 < \infty$. Define

$$T_n = \frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}},$$

with $T_n = 0$ if the denominator is zero.

(a) Prove that $T_n \xrightarrow{d} N(0,1)$, naming every limit theorem used. [7]

(b) If instead $\mathbb{E}X_1 = \mu \neq 0$ and $\mathbb{E}X_1^2 < \infty$, determine the behavior of $T_n/\sqrt{n}$. [3]

Problem 3 — 15 points

Let $X_1, \ldots, X_n$ be i.i.d. geometric random variables counting failures before the first success:

$$\mathbb{P}_p(X_i = x) = p(1-p)^x, \qquad x = 0, 1, \ldots, \quad 0 < p < 1,$$

where $n \geq 2$. Extend the model to $p = 1$ by the degenerate law $\mathbb{P}_1(X_i = 0) = 1$, and let $T = \sum_i X_i$.

(a) Prove that T is sufficient and complete for p. A statement that this is an exponential family is not, by itself, a proof of completeness. [5]

(b) Find an unbiased function of T estimating p and prove that it is the UMVU estimator. [6]

(c) Find the MLE of p on the parameter space $(0, 1]$ and derive its asymptotic distribution when $0 < p < 1$. [4]

Problem 4 **15 points**

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p), where the parameter space is the closed interval $[0, 1]$, and let $\widehat{p}_n = \bar{X}$.

(a) For a fixed $p \in (0, 1)$, derive the limiting distribution of $\widehat{p}_n$. [4]

(b) Determine the exact distribution of $\widehat{p}_n$ at $p = 0$ and explain precisely why the usual interior-parameter asymptotic-normality theorem does not apply there. [4]

(c) Let the true parameter vary with n as $p_n = \lambda/n$, where $\lambda > 0$. Find the limits in distribution of

$$n\widehat{p}_n \quad \text{and} \quad n(\widehat{p}_n - p_n).$$

Explain why the $\sqrt{n}$ scaling is not informative in this local regime. [7]

Problem 5 **15 points**

Let $X_1, \ldots, X_n$ be i.i.d. from the logistic location family

$$f_\theta(x) = \frac{e^{-(x-\theta)}}{(1 + e^{-(x-\theta)})^2}, \qquad x \in \mathbb{R}, \quad \theta \in \mathbb{R}.$$

Write $F_\theta(x) = \{1 + e^{-(x-\theta)}\}^{-1}$.

(a) Write the log-likelihood, derive the score equation, and prove that the MLE exists and is unique for every sample. [5]

(b) Compute the Fisher information in one observation and derive the asymptotic distribution of the MLE, stating the regularity facts used. [5]

(c) Let M_n be the sample median for odd n. Derive its asymptotic distribution and its asymptotic relative efficiency with respect to the MLE under squared-error comparison. [5]

Problem 6 **15 points**

Consider the possibly rank-deficient normal linear model

$$Y = X\beta + \varepsilon, \qquad \varepsilon \sim N_n(0, \sigma^2 I_n),$$

where X is a known $n \times p$ matrix of rank $r < p$, $n > r$, and $\beta \in \mathbb{R}^p$. Let $\tau = c^\top \beta$.

(a) Prove that τ is estimable by a linear unbiased estimator if and only if $c \in \mathcal{C}(X^\top)$. [5]

(b) Assuming this condition, use the Moore–Penrose inverse X^+ to construct the BLUE of τ, find its variance, and prove its optimality. [6]

(c) With $s^2 = \|(I - XX^+)Y\|^2/(n - r)$, construct an exact $(1 - \alpha)$ confidence interval for τ. [4]

Problem 7 **10 points**

Let $X_1, \ldots, X_n$ be i.i.d. $N(0, v)$, where $v > 0$. Consider $H_0 : v = v_0$ against $H_1 : v \neq v_0$.

(a) Derive the MLE of v and the generalized likelihood ratio statistic. Express $-2 \log \Lambda$ as a function of $T/(nv_0)$, where $T = \sum_i X_i^2$. [5]

(b) Give an exact level-α GLR rejection rule using the χ^2_n distribution, and give the asymptotic level-α rule from Wilks' theorem. State why Wilks' theorem applies. [5]

Problem 8 **10 points**

Suppose $Y \mid \theta \sim N(\theta, \sigma^2)$, where $\sigma > 0$ is known, and put the Laplace prior

$$\pi(\theta) = \frac{\lambda}{2} e^{-\lambda|\theta|}, \qquad \lambda > 0.$$

(a) Write the posterior density up to its normalizing constant. [2]

(b) Derive the posterior mode for every observed y, including the threshold cases. [6]

(c) Explain why this posterior mode is generally not the Bayes estimator under squared-error loss. [2]