Probability and Statistics

PhD Qualification Exam

Mock Examination No. 2

Total: 100 points

- There are eight compulsory problems. The point values sum to 100.
- Give all necessary details. Unsupported final answers receive little credit.
- You may quote standard theorems only after verifying their assumptions.

Problem 1 **10 points**

Let U and V be independent Uniform$(0, 1)$ random variables. Define

$$M = \max\{U, V\}, \qquad R = \frac{\min\{U, V\}}{\max\{U, V\}},$$

where the ratio is defined arbitrarily on the null event $\{M = 0\}$.

(a) Derive the joint density of (M, R), including its support. [5]

(b) Find the marginal laws of M and R, and determine whether they are independent. [3]

(c) Compute $\mathbb{E}(\min\{U, V\} \mid M)$. [2]

Problem 2 **10 points**

Let $(A_n)_{n\geq 1}$ be independent events with $\mathbb{P}(A_n) = 1/n$. For a fixed $\gamma > 0$, set

$$X_n = n^\gamma \mathbf{1}_{A_n}.$$

(a) Determine whether $X_n \to 0$ in probability and whether $X_n \to 0$ almost surely. Prove both assertions. [5]

(b) For each $p > 0$, determine exactly when $X_n \to 0$ in L^p. What happens to $\mathbb{E}X_n$ in the three cases $\gamma < 1$, $\gamma = 1$, and $\gamma > 1$? [5]

Problem 3 **15 points**

A fair coin is tossed repeatedly. Let T be the first time at which the consecutive pattern HHT has appeared.

(a) Construct a Markov chain whose state records the longest suffix of the observations that is also a prefix of HHT. Give its transition matrix, including the absorbing state. [5]

(b) Starting with no observations, find the probability generating function $\mathbb{E}[s^T]$ for $0 \leq s < 1$. [6]

(c) Compute $\mathbb{E}T$ from the generating function. [4]

Problem 4 **15 points**

Let $(N_t)_{t\geq 0}$ be a Poisson process of rate $\lambda > 0$, and let $0 < T_1 < \cdots < T_{N_T} < T$ denote its arrival times in the interval $(0, T)$.

(a) State and prove the conditional joint density of $(T_1, \ldots, T_n)$ given $N_T = n$. [5]

(b) Given $N_T = n$, find the probability that every two consecutive arrivals are at least δ apart. Treat all possible values of $\delta \geq 0$ and n. [6]

(c) For $0 \leq \delta \leq T$, compute the expected number of unordered pairs of arrivals in $(0, T)$ whose time separation is at most δ. [4]

Problem 5 **15 points**

Let $(B_t)_{t\geq 0}$ be standard Brownian motion and

$$\tau_a = \inf\{t \geq 0 : |B_t| = a\}, \qquad a > 0.$$

(a) Prove that $\tau_a < \infty$ almost surely and, for $\lambda > 0$, derive

$$\mathbb{E}e^{-\lambda\tau_a} = \frac{1}{\cosh(a\sqrt{2\lambda})}.$$

[5]

(b) Compute $\mathbb{E}\tau_a$, $\mathbb{E}\tau_a^2$, and $\mathrm{Var}(\tau_a)$. Justify the passage from the Laplace transform to the moments. [5]

(c) Prove that τ_a and $\mathbf{1}_{\{B_{\tau_a}=a\}}$ are independent, and identify the law of the indicator. [5]

Problem 6 **15 points**

Let $S = \{0, 1, \ldots, m\}$, where $m \geq 2$, and let the target mass function be

$$\pi_i = \frac{2(i+1)}{(m+1)(m+2)}, \qquad i \in S.$$

Use the nearest-neighbor proposal that moves left or right with probability $1/2$ in the interior and replaces the unavailable move at each endpoint by a self-proposal. Apply the Metropolis acceptance probability $\min(1, \pi_j/\pi_i)$.

(a) Write every nonzero transition probability of the resulting Markov chain. [5]

(b) Prove irreducibility, aperiodicity, and reversibility, and identify its unique stationary distribution. [6]

(c) If $\tau_0^+ = \inf\{n \geq 1 : X_n = 0\}$, compute $\mathbb{E}_0\tau_0^+$ and state the limiting value of $P^n(i, j)$. [4]

Problem 7 **10 points**

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p) and let the parameter of interest be the odds

$$\theta = \frac{p}{1-p} \in (0, \infty).$$

(a) Find the MLE of θ, including the boundary samples $\sum_i X_i = 0$ and $\sum_i X_i = n$ when the parameter space is replaced by its closure. [3]

(b) For fixed $0 < p < 1$, derive the asymptotic distribution of the MLE of θ. [4]

(c) Construct an asymptotic $(1 - \alpha)$ confidence interval for θ by first constructing a Wald interval for $\log \theta$. [3]

Problem 8 **10 points**

Independently observe

$$X \sim \text{Poisson}(t_1\lambda_1), \qquad Y \sim \text{Poisson}(t_2\lambda_2),$$

where $t_1, t_2 > 0$ are known. Consider $H_0 : \lambda_1 = \lambda_2$ against $H_1 : \lambda_1 > \lambda_2$.

(a) Find the conditional distribution of X given $S = X + Y = s$ under H_0, and show that it is free of the nuisance rate. [3]

(b) Construct a test whose conditional size given every $S = s$ is exactly α, allowing randomization at one boundary point. [4]

(c) Prove, using monotone likelihood ratios, that the test is uniformly most powerful among tests having conditional size α for every s. [3]