Probability and Statistics

PhD Qualification Exam

Mock Examination No. 1

Total: 100 points

- There are eight compulsory problems. The point values sum to 100.
- Give all necessary details. Unsupported final answers receive little credit.
- You may quote standard theorems only after verifying their assumptions.

Problem 1 **10 points**

Let $X \sim \mathrm{Exp}(\lambda)$ and $Y \sim \mathrm{Exp}(\mu)$ be independent, where $\lambda, \mu > 0$. Define

$$T = \min\{X, Y\}, \qquad I = \mathbf{1}_{\{X<Y\}}, \qquad R = |X - Y|.$$

Ties have probability zero.

(a) Find the joint law of (T, I) and prove that T and I are independent. [4]

(b) Find the conditional law of R given $I = 1$ and given $I = 0$. [4]

(c) Compute $\mathbb{E}R$. [2]

Problem 2 **15 points**

Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}X_1 = \mu$, $\mathrm{Var}(X_1) = \sigma^2$, and $0 < \sigma^2 + \mu^2 < \infty$. For each n, let $N_n \sim \mathrm{Poisson}(n)$ be independent of the sequence, and set $S_n = \sum_{k=1}^{N_n} X_k$, with an empty sum equal to zero.

(a) Compute $\mathbb{E}S_n$ and $\mathrm{Var}(S_n)$. [3]

(b) Prove that

$$\frac{S_n - n\mu}{\sqrt{n(\sigma^2 + \mu^2)}} \xrightarrow{d} N(0, 1).$$

State precisely which characteristic-function theorem you use. [8]

(c) Suppose in addition that $\mu = 0$. With the ratio defined as zero on $\{N_n = 0\}$, prove that

$$\frac{S_n}{\sigma\sqrt{N_n}} \xrightarrow{d} N(0, 1).$$

[4]

Problem 3 **10 points**

An urn initially contains $a \geq 1$ red balls and $b \geq 1$ blue balls. At each step a ball is drawn uniformly, returned, and one additional ball of the same color is added. Let R_n be the number of red balls after n draws, $c = a + b$, and

$$Z_n = \frac{R_n}{c+n}.$$

(a) Prove that (Z_n) is a bounded martingale and hence converges almost surely and in L^1 to a random variable Z_∞. [4]

(b) Compute $\mathrm{Var}(Z_\infty)$ and conclude that Z_∞ is not almost surely constant. [6]

Problem 4 **15 points**

Let $(B_s)_{s\geq 0}$ be standard Brownian motion and $M_t = \max_{0\leq s\leq t} B_s$, where $t > 0$ is fixed. Write

$$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}.$$

(a) For $a > 0$ and $b < a$, use the reflection principle to prove

$$\mathbb{P}(M_t \geq a,\ B_t \leq b) = \mathbb{P}(B_t \geq 2a - b).$$

[5]

(b) Derive the joint density of (M_t, B_t) on its support. [6]

(c) Prove that $M_t - B_t \overset{d}{=} |B_t|$ and compute $\mathbb{E}(M_t - B_t)$. [4]

Problem 5 **15 points**

For a fixed $0 < r < 1$, let $X_1, \ldots, X_n$ be i.i.d. with density

$$f_\theta(x) = \{1 + \theta(2x - 1)\}\mathbf{1}_{(0,1)}(x), \qquad \theta \in [-r, r].$$

Assume the true parameter satisfies $|\theta| < r$.

(a) Find a method-of-moments estimator, project it onto $[-r, r]$, and derive its limiting distribution. [5]

(b) Write the likelihood and log-likelihood. Prove strict concavity with probability one and characterize the MLE, including possible boundary cases. [4]

(c) Compute the Fisher information in one observation and derive the limiting distribution of the MLE. Compare its asymptotic variance with that of the method-of-moments estimator. [6]

Problem 6 **10 points**

Let X and Y be independent Poisson(λ) random variables, where $\lambda > 0$, and define $U = (X - Y)^2/2$ and $T = X + Y$.

(a) Show that U is unbiased for λ, and show that T is complete and sufficient for λ. [3]

(b) Compute $\mathbb{E}(U \mid T)$ and identify the UMVU estimator of λ. [4]

(c) Compute the variances of U and its Rao–Blackwell improvement. [3]

Problem 7 **15 points**

Consider the normal linear model

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \qquad i = 1, \ldots, n, \qquad n \geq 3,$$

where the x_i are fixed, $\sum_i x_i = 0$, $S_{xx} = \sum_i x_i^2 > 0$, and the errors are i.i.d. $N(0, \sigma^2)$. Let $s^2 = \mathrm{RSS}/(n-2)$.

(a) Derive the least-squares estimators and their joint distribution. State the distribution of RSS/σ^2 and the relevant independence. [5]

(b) Prove directly that the least-squares estimator of β_1 has minimum variance among all linear unbiased estimators of β_1. [5]

(c) Give an exact $(1-\alpha)$ confidence interval for β_1 and an exact $(1-\alpha)$ prediction interval for a future response at a fixed covariate x_0. [5]

Problem 8 **10 points**

Let $X \mid p \sim \text{Binomial}(m, p)$ and let the prior be $p \sim \text{Beta}(a, b)$, where $a, b > 1$. For an observed value $X = x$, consider the loss

$$L(p, d) = \frac{(d-p)^2}{p(1-p)}, \qquad 0 < d < 1.$$

(a) Find the posterior distribution of p. [3]

(b) Derive the Bayes estimator under the stated loss, checking that all posterior expectations used are finite. [5]

(c) Determine when this estimator equals the posterior mean. [2]