Mock Qualifying Examination 10

Computational and Applied Mathematics

Total: 100 points

Instructions. Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

Problem 1. [10 points] Let $\mathcal{F}$ be the class of continuous strictly increasing functions $f : [0,1] \to \mathbb{R}$ satisfying $f(0) < 0 < f(1)$. A deterministic algorithm may choose k points adaptively and, at each point, receives only the sign of f. It must then return an interval guaranteed to contain the zero of every function in $\mathcal{F}$ consistent with the answers. Determine the smallest possible worst-case length of this interval, and prove your result. The endpoint signs are known without a query; a query returning zero may terminate the algorithm.

Problem 2. [20 points] Let $f \in C^2[0,1]$ satisfy $f''(x) \geq m > 0$. For a partition

$$\mathcal{T}: \quad 0 = x_0 < x_1 < \cdots < x_n = 1,$$

let $I_{\mathcal{T}} f$ be the continuous piecewise-linear interpolant of f, and define

$$E_n = \inf_{\mathcal{T}} \|f - I_{\mathcal{T}} f\|_\infty ,$$

where the infimum is over partitions with exactly n subintervals. Determine the sharp leading asymptotic behavior of E_n as $n \to \infty$, and construct a family of asymptotically optimal partitions. You may use the standard pointwise remainder formula for linear interpolation.

Problem 3. [20 points] Let $A \in \mathbb{R}^{m\times n}$ have full column rank. A thin QR factorization $A = QR$, the vector $c = Q^T b$, and the old minimum residual norm $\eta = \|b - Qc\|_2$ are available. One new observation

$$a^T x \approx \beta$$

is appended to the least-squares problem. Without recomputing a factorization of the enlarged matrix, compute the new least-squares solution and its minimum residual norm in $O(n^2)$ operations. Prove the update and the complexity claim; only the compressed state needs to be updated.

Problem 4. [15 points] Consider

$$-u'' = f \quad \text{on } (0,1), \qquad -u'(0) = g_0, \quad -u'(1) = g_1, \qquad \int_0^1 u(x)\,dx = 0,$$

and assume the normalized solution belongs to $C^3[0,1]$. Let $h = 1/N$, $x_j = (j - \frac{1}{2})h$, and

$$\bar{f}_j = \frac{1}{h}\int_{(j-1)h}^{jh} f(x)\,dx.$$

For unknown cell values U_j, introduce face fluxes by

$$F_{1/2} = g_0, \qquad F_{N+1/2} = g_1, \qquad F_{j+1/2} = -\frac{U_{j+1} - U_j}{h} \quad (1 \leq j < N),$$

and impose

$$\frac{F_{j+1/2} - F_{j-1/2}}{h} = \bar{f}_j, \qquad h\sum_{j=1}^{N} U_j = 0.$$

Analyze this discrete problem completely: determine when it is solvable, characterize its solutions before normalization, construct the normalized solution in $O(N)$ operations, and determine its convergence order at the cell centers.

Problem 5. [15 points] Let $a(x) > 0$ be smooth and periodic, and consider

$$u_t + (a(x)u)_x = 0.$$

On a periodic uniform grid, with $a_j = a(x_j)$ and $\lambda = \Delta t/h \geq 0$, consider

$$U_j^{n+1} = U_j^n - \lambda\big(a_j U_j^n - a_{j-1}U_{j-1}^n\big).$$

Determine its consistency order and a sharp condition for positivity preservation. Identify a mesh norm, uniformly equivalent to the maximum norm, in which the update is contractive, and characterize all steady grid states that it preserves exactly. A scheme is called well balanced for a family of states if one update leaves every state in that family unchanged.

Problem 6. [20 points; choose Problem 6 or Problem 7] For $0 < \varepsilon < \frac{1}{2}$, consider

$$x' = -x, \qquad \varepsilon y' = x - y, \qquad x(0) = 1, \quad y(0) = b.$$

Obtain an approximation to $y(t)$ through terms of order ε that is uniformly valid for all $t \geq 0$, with a uniform $O(\varepsilon^2)$ error bound. Write the exact solution as the sum of a term proportional to e^{-t} and a remainder $r_\varepsilon(t)$, and define

$$t_\varepsilon = \inf\{t \geq 0 : |r_\varepsilon(t)| \leq \varepsilon\}.$$

For fixed $b \neq 1$, determine the asymptotic behavior of t_ε. Explain what changes when $b = 1$, and determine the ε-dependent initial value for which $r_\varepsilon(t) \equiv 0$.

Problem 7. [20 points; choose Problem 6 or Problem 7] Let $A \in \mathbb{R}^{m\times n}$ have full row rank and suppose

$$P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$$

is bounded and contains a strictly positive point. For $\mu > 0$, let $x(\mu)$ minimize

$$c^T x - \mu \sum_{i=1}^{n} \log x_i \qquad \text{subject to } Ax = b, \quad x > 0.$$

Prove that $x(\mu)$ exists and is unique. Show that it determines a dual-feasible certificate, obtain an explicit primal–dual gap, and use the gap to prove that every limit point of $x(\mu)$ as $\mu \downarrow 0$ is primal optimal.