Mock Qualifying Examination 09

Computational and Applied Mathematics

Total: 100 points

Instructions. Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

Problem 1. [10 points] Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad y' = Jy,$$

and let $y(0) = y_0 \neq 0$. Apply the explicit Euler method with step size $h > 0$. Determine the exact evolution of the numerical amplitude. Explain why the method converges on every fixed time interval, and identify the first long-time scale on which the numerical amplitude error is no longer small. Give the limiting amplification factor on that scale.

Problem 2. [20 points] Let $n \geq 1$, let $x_0, \ldots, x_n$ be distinct real nodes, and let $p \in \mathcal{P}_n$ satisfy $p(x_j) = f_j$, and define

$$\omega(x) = \prod_{j=0}^{n} (x - x_j), \qquad w_j = \frac{1}{\omega'(x_j)}.$$

Derive a formula that evaluates $p(x)$ in $O(n)$ operations without first forming its monomial coefficients, including the case $x = x_j$. A new node x_* and value f_* are then added. Show how all weights can be updated in $O(n)$ operations. Do the same when one existing node is deleted, and prove the formulas you use.

Problem 3. [20 points] Let $A^{(0)} = A$ be a real symmetric $n \times n$ matrix, where $n \geq 2$. At step k, choose $p < q$ so that $|a_{pq}^{(k)}|$ is maximal among the off-diagonal entries, and choose a plane rotation Q_k for which

$$A^{(k+1)} = Q_k^T A^{(k)} Q_k$$

has zero (p, q) and (q, p) entries. Prove a dimension-explicit geometric bound for the off-diagonal Frobenius norm. Deduce that the off-diagonal part converges to zero, and prove that the sorted diagonal entries converge to the sorted eigenvalues of A. Include the case in which the iteration reaches a diagonal matrix in finitely many steps.

Problem 4. [15 points] Consider

$$-u'' = f \quad \text{on } (0, 1), \qquad u(0) = u(1) = 0,$$

with $u \in C^6[0, 1]$, on the uniform grid $x_j = jh$, $h = 1/N$. Without changing the standard tridiagonal centered-difference matrix or introducing ghost values, modify only the sampled right-hand side, using f at neighboring grid points, to obtain a fourth-order method. Derive the method and prove both its local consistency order and its global maximum-norm convergence order.

Problem 5. [15 points] For

$$u_t = u_{xx} - u^3, \qquad 0 < x < 1, \qquad u(0,t) = u(1,t) = 0,$$

on the grid $x_j = jh$, $h = 1/N$, consider the fully implicit finite-difference scheme

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{h^2} - (U_j^{n+1})^3, \qquad 1 \le j \le N-1,$$

with $U_0^n = U_N^n = 0$ and $k > 0$. Prove that every time step has a unique solution for arbitrary k and h. Find a nonincreasing discrete analogue of the continuous energy, and determine the local consistency order for a sufficiently smooth exact solution.

Problem 6. [20 points; choose Problem 6 or Problem 7] Let $f \in C^2[0,1]$ and

$$I(\varepsilon) = \int_0^1 \frac{f(x)}{x^2 + \varepsilon^2}\, dx, \qquad \varepsilon > 0.$$

Obtain an asymptotic expansion of $I(\varepsilon)$ through its bounded term, with an $O(\varepsilon)$ remainder. Discuss degeneracies in the leading terms and justify the remainder estimate.

Problem 7. [20 points; choose Problem 6 or Problem 7] Given $y \in \mathbb{R}^n$, consider

$$\min_{x_1 \le x_2 \le \cdots \le x_n} \frac{1}{2} \sum_{i=1}^{n} (x_i - y_i)^2.$$

Characterize the unique minimizer and give an $O(n)$-operation algorithm for computing it. Prove its correctness and complexity.