Mock Qualifying Examination 08

Computational and Applied Mathematics

Total: 100 points

Instructions. Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

Problem 1. [10 points] Let $A = A^T \in \mathbb{R}^{n\times n}$ have eigenvalues ordered so that $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$, with $\lambda_1 < 0$. Let v_1 be a unit eigenvector for λ_1, and suppose $v_1^T x_0 \neq 0$. Define

$$x_k = \frac{A^k x_0}{\|A^k x_0\|_2}.$$

Do the vectors x_k converge? Prove what happens to the one-dimensional subspaces they span, and determine the convergence order of those subspaces.

Problem 2. [20 points] Let $M > 0$, $\delta > 0$, and

$$\mathcal{F}_M = \{f \in C^3[-1,1] : \|f'''\|_\infty \leq M\}.$$

For a chosen $0 < h \leq 1$, the only available data are Y_-, Y_0, Y_+ satisfying

$$|Y_- - f(-h)| \leq \delta, \qquad |Y_0 - f(0)| \leq \delta, \qquad |Y_+ - f(h)| \leq \delta.$$

Among all linear estimators $aY_- + bY_0 + cY_+$ and all $0 < h \leq 1$, determine the smallest possible worst-case error for estimating $f'(0)$. Give the optimal coefficients and step size, including the case in which the constraint $h \leq 1$ is active. Prove that your bound is sharp.

Problem 3. [20 points] Let $A \in \mathbb{R}^{m\times n}$ have full column rank, assume that $Ax = b$ is consistent, and assume that every row a_i^T, $1 \leq i \leq m$, is nonzero. Starting from any x_0, independently choose an index i_k at step k with

$$\mathbb{P}(i_k = i) = \frac{\|a_i\|_2^2}{\|A\|_F^2},$$

and set

$$x_{k+1} = x_k + \frac{b_{i_k} - a_{i_k}^T x_k}{\|a_{i_k}\|_2^2}\, a_{i_k}.$$

Derive a quantitative mean-square convergence bound, including the dependence on the singular values of A, and, for $0 < \varepsilon < 1$, give the iteration count sufficient to reduce the expected squared error by a factor ε^2. Give the dense arithmetic cost after preprocessing, and exhibit a family for which your convergence factor is attained exactly. Treat any degenerate endpoint in your bound separately.

Problem 4. [15 points] Let $u(x) = \phi(r)$, $r = |x|$, be a smooth radial solution in the unit ball in $\mathbb{R}^d$, $d \geq 2$, of

$$-\left(\phi''(r) + \frac{d-1}{r}\phi'(r)\right) = f(r), \qquad 0 < r < 1, \qquad \phi(1) = 0.$$

On the grid $r_j = jh$, the expression in parentheses is discretized by centered differences at $r_j > 0$. Complete the finite-difference scheme at $r = 0$ using only U_0 and U_1. Prove its consistency order there and explain why treating the origin as an ordinary one-dimensional Neumann endpoint gives an incorrect equation when $d > 1$.

Problem 5. **[15 points]** The differential equation $u_{tt} = c^2 u_{xx}$ on $x \le L$ has the exact outgoing condition $u_t + cu_x = 0$ at $x = L$. Let $x_j = L + (j - N)h$ and suppose this condition is replaced first by

$$\dot{U}_N + c\frac{U_N - U_{N-1}}{h} = 0.$$

For a harmonic field sampled from the continuous wave equation,

$$U_j(t) = e^{-ic\kappa t}\left(e^{i\theta(j-N)} + R(\theta)e^{-i\theta(j-N)}\right), \qquad \theta = \kappa h, \quad 0 < \theta < \pi,$$

determine the reflected-to-incident amplitude $R(\theta)$ and its long-wave order. Repeat the leading-order calculation when the spatial derivative in the boundary condition is replaced by

$$\frac{3U_N - 4U_{N-1} + U_{N-2}}{2h}.$$

Explain the relation between the two reflection orders and the consistency orders of the boundary formulas.

Problem 6. **[20 points; choose Problem 6 or Problem 7]** Fix $0 < a < 1$. For $\varepsilon > 0$ define

$$x_{k+1} = x_k + \varepsilon + x_k^2, \qquad x_0 = -a,$$

and let N_ε be the first index for which $x_{N_\varepsilon} \ge a$. Prove that N_ε is finite and determine its leading asymptotic behavior as $\varepsilon \downarrow 0$.

Problem 7. **[20 points; choose Problem 6 or Problem 7]** In a zero-sum matrix game the row player maximizes and the column player minimizes $p^T Aq$, where p, q are probability vectors and

$$A = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 3 \\ 1 & -3 & 0 \end{pmatrix}.$$

Formulate the two security problems as a primal–dual pair of linear programs. Determine the value of the game and all optimal mixed strategies, and prove uniqueness or nonuniqueness as appropriate.