Mock Qualifying Examination 07

Computational and Applied Mathematics

Total: 100 points

Instructions. Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

Problem 1. [10 points] Let $p > 1$, $0 < x_0 < 1$, and

$$x_{k+1} = x_k - x_k^p.$$

Prove convergence and determine a sharp asymptotic equivalent for x_k as $k \to \infty$. Classify the convergence as q-linear or sublinear.

Problem 2. [20 points] Let $P_0, \dots, P_n$ be the Legendre polynomials normalized by

$$P_k(1) = 1, \qquad \int_{-1}^{1} P_k(x)P_\ell(x)\,dx = \frac{2}{2k+1}\delta_{k\ell}.$$

For a fixed $\xi \in [-1, 1]$, determine the unique solution of

$$\min\left\{\|p\|_{L^2(-1,1)} : p \in \mathcal{P}_n,\ p(\xi) = 1\right\}$$

and the minimum value. Specialize your result to $\xi = 1$ and find the sharp constant C_n in

$$|p(1)| \le C_n \|p\|_{L^2(-1,1)}, \qquad p \in \mathcal{P}_n.$$

Problem 3. [20 points] For

$$p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0,$$

let

$$C = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -c_0 & -c_1 & \cdots & -c_{n-1} \end{pmatrix}.$$

Suppose λ is a simple real root of p. Under real coefficient perturbations $c_j \mapsto c_j + \varepsilon d_j$, derive the first-order change of the nearby root and the sharp absolute condition number for $\|d\|_2 \le 1$. Recover the same sensitivity from the eigenvalue problem for C. Explain, with an example, what fails when the root is multiple.

Problem 4. [15 points] On a nonuniform grid let $h_j = x_{j+1} - x_j$ and consider

$$(\mathcal{D}_2 U)_j = \frac{2}{h_{j-1} + h_j}\left(\frac{U_{j+1} - U_j}{h_j} - \frac{U_j - U_{j-1}}{h_{j-1}}\right).$$

Determine its pointwise consistency order on a quasiuniform family of meshes. Here quasiuniform means that, independently of refinement,

$$\frac{\max_j h_j}{\min_j h_j} \le C$$

for a fixed constant C. Give a local mesh condition that restores second order, and show by an explicit quasiuniform mesh family and a smooth function that quasiuniformity alone does not suffice.

Problem 5. [15 points] On the infinite grid $x_j = jh$, consider the explicit heat update

$$U_j^{n+1} = rU_{j-1}^n + (1-2r)U_j^n + rU_{j+1}^n, \qquad j \in \mathbb{Z}, \qquad r \in \mathbb{R}.$$

Characterize, with sharpness, all r for which the update preserves nonnegativity and is a contraction in ℓ^∞. Suppose

$$\sum_{j\in\mathbb{Z}} (1+j^2)|U_j^0| < \infty.$$

Determine exactly how the first three discrete spatial moments

$$M_\ell^n = h \sum_{j\in\mathbb{Z}} x_j^\ell U_j^n, \qquad \ell = 0, 1, 2,$$

evolve. Interpret the result when $r = k/h^2$ and $U^0 \geq 0$.

Problem 6. [20 points; choose Problem 6 or Problem 7] For fixed $V > 0$ and $0 < \varepsilon \ll 1$, a nondimensional projectile satisfies

$$z' = v, \qquad v' = -1 - \varepsilon v|v|, \qquad z(0) = 0, \quad v(0) = V.$$

Let $T > 0$ be its first return time to $z = 0$. Through first order in ε, determine the maximum height, the time to maximum height, the total flight time, and the impact speed. Justify the matching at the apex and state the parameter regime in which your expansions are valid.

Problem 7. [20 points; choose Problem 6 or Problem 7] Given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $\rho > 0$, consider

$$\min_{x\in\mathbb{R}^n} \max_{\|\Delta\|_F \leq \rho} \|(A+\Delta)x - b\|_2 .$$

Reduce this problem to an explicit deterministic convex optimization problem and prove the equivalence, including all zero cases. Prove existence of a minimizer, and, when $b \neq 0$, give a necessary and sufficient condition for $x = 0$ to be optimal. Do not assume that the minimizer is unique.