Mock Qualifying Examination 06

Computational and Applied Mathematics

Total: 100 points

Instructions. Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

Problem 1. [10 points] Let $f \in C^2$ near a simple zero α. Starting from x_k, choose s_k so that

$$f'(x_k)s_k = -f(x_k) + r_k, \qquad 0 \le \eta_k, \quad |r_k| \le \eta_k |f(x_k)|,$$

and set $x_{k+1} = x_k + s_k$. Establish a local error estimate that determines the convergence rate when, respectively, $\sup_k \eta_k < 1$, $\eta_k \to 0$, and $\eta_k = O(|f(x_k)|)$. Give an example showing what may happen at the endpoint $\eta_k = 1$.

Problem 2. [20 points] Fix $N \ge 1$. At distinct nodes $x_0, \ldots, x_N \in [-1, 1]$, let $I_N f$ be the polynomial interpolant of degree at most N, and define

$$\mathcal{E}(x_0, \ldots, x_N) = \sup_{\substack{f \in C^{N+1}[-1,1] \\ \|f^{(N+1)}\|_\infty \le 1}} \|f - I_N f\|_\infty .$$

Determine the nodes that minimize $\mathcal{E}$, and find the minimum value. Your proof must include both the worst-case error for fixed nodes and the optimality of the chosen nodes.

Problem 3. [20 points] Let $H \in \mathbb{R}^{n\times n}$ be symmetric positive definite and let $B \in \mathbb{R}^{m\times n}$ have full row rank, where $1 \le m \le n$. Consider

$$K = \begin{pmatrix} H & B^T \\ B & 0 \end{pmatrix}.$$

Determine the inertia of K and prove that K is nonsingular. Then give an algorithm for solving $K(x^T, \lambda^T)^T = (f^T, g^T)^T$ that uses only positive-definite systems of orders n and m. Justify why the smaller system in your algorithm is positive definite and state the dense arithmetic cost.

Problem 4. [15 points] For $u_t + a u_x = 0$, $a > 0$, on the unit periodic interval, let $h = 1/J$ and write

$$\frac{a\tau}{h} = q + \theta, \qquad q \in \{0, 1, 2, \ldots\}, \quad 0 \le \theta < 1.$$

With indices understood modulo J, consider

$$U_j^{n+1} = (1-\theta)U_{j-q}^n + \theta U_{j-q-1}^n.$$

Prove preservation of the discrete mass and contractivity in both the discrete ℓ^1 and ℓ^∞ norms, with no restriction on $a\tau/h$. For smooth exact data at $t = 0$, derive a maximum-norm error bound up to a fixed time T. Interpret your bound, including the case $\theta = 0$.

Problem 5. [15 points] The wave equation $u_{tt} = c^2 u_{xx}$ is discretized in space on a periodic grid by

$$\ddot{U}_j(t) = c^2 \frac{U_{j+1}(t) - 2U_j(t) + U_{j-1}(t)}{h^2}.$$

For resolved wave numbers $0 \le \kappa \le \pi/h$, determine the numerical dispersion relation, phase velocity, and group velocity (defined as $d\omega_h/d\kappa$) on the right-going branch. Interpret the behavior near the Nyquist wave number. Finally, reconcile that behavior with the formal order of spatial consistency.

Problem 6. [20 points; choose Problem 6 or Problem 7] For $0 < \varepsilon \ll 1$, consider

$$\varepsilon y'' + xy' = 0, \qquad -1 < x < 1, \qquad y(-1) = -1, \quad y(1) = 1.$$

Determine the limiting outer behavior, locate the transition region and its scale, and obtain a leading profile that is uniform on $[-1, 1]$. Justify the claimed uniform accuracy and explain why an ordinary power-series expansion in ε misses the transition.

Problem 7. [20 points; choose Problem 6 or Problem 7] Let

$$\mathcal{B}_n = \{X \in \mathbb{R}^{n\times n} : X_{ij} \geq 0,\ X\mathbf{1} = \mathbf{1},\ X^T\mathbf{1} = \mathbf{1}\}.$$

Characterize all extreme points of $\mathcal{B}_n$. Hence, for an arbitrary $C \in \mathbb{R}^{n\times n}$, give an equivalent finite optimization problem for

$$\max_{X\in\mathcal{B}_n} \sum_{i,j=1}^{n} C_{ij}X_{ij}.$$

Your proof must not assume the claimed characterization of the extreme points.