Mock Qualifying Examination 05

Computational and Applied Mathematics

Total: 100 points

Instructions. Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

Problem 1. [10 points] Let $f \in C^3$ have a simple zero α, with $f''(\alpha) \neq 0$. Assume that two distinct, sufficiently close starting values generate well-defined secant iterates converging to α. Without quoting the known order of the secant method, derive its leading two-step error relation and determine its local order of convergence.

Problem 2. [20 points] Let f be continuous and 2π-periodic. If $S_k f$ is its kth Fourier partial sum, define

$$\sigma_N f = \frac{1}{N+1}\sum_{k=0}^{N} S_k f.$$

Prove that $\sigma_N f \to f$ uniformly. Also prove that $f \geq 0$ implies $\sigma_N f \geq 0$ and that

$$\|\sigma_N f\|_\infty \leq \|f\|_\infty .$$

If f is Lipschitz, obtain the uniform error bound $O((\log N)/N)$. Do not quote an approximate-identity theorem without verifying its hypotheses in this setting.

Problem 3. [20 points] Let $T \in \mathbb{R}^{n\times n}$ be real symmetric tridiagonal, with diagonal entries α_i and nonzero subdiagonal entries β_i. For a real number μ, define

$$p_0 = 1, \qquad p_1 = \alpha_1 - \mu, \qquad p_k = (\alpha_k - \mu)p_{k-1} - \beta_{k-1}^2 p_{k-2}.$$

Assume $p_k(\mu) \neq 0$ for $k = 1, \dots, n$. Prove that the number of eigenvalues of T below μ equals the number of sign changes in $p_0(\mu), p_1(\mu), \dots, p_n(\mu)$. Explain how this gives an algorithm for locating a specified eigenvalue to absolute accuracy δ, and give its complexity when an initial bracket $[L, U]$ is known.

Problem 4. [15 points] For $a > 0$, consider the advection equation

$$u_t + a u_x = 0, \qquad 0 < x < 1, \qquad u(0,t) = 0.$$

On the grid $x_j = jh$, $h = 1/N$, consider

$$\dot U_j + a\frac{U_{j+1} - U_{j-1}}{2h} = 0, \quad 1 \leq j < N, \qquad U_0 = 0,$$

together with the outflow equation

$$\dot U_N + a\frac{U_N - U_{N-1}}{h} = 0.$$

Prove an h-uniform discrete L^2 stability estimate. Determine the local consistency orders in the interior and at the outflow point for a sufficiently smooth exact solution. Explain why the last equation does not impose an additional boundary condition on the continuous problem.

Problem 5. [15 points] Two adjacent grid points lie on opposite sides of a material interface located at their midpoint. The diffusion coefficient is $a_- > 0$ on the left half-cell and $a_+ > 0$ on the right half-cell. Determine the unique coefficient $\widehat{a}$ for which

$$J_{j+1/2} = -\widehat{a}\,\frac{U_{j+1} - U_j}{h}$$

is exact for every continuous piecewise-linear profile whose physical flux is continuous across the interface. Show that the resulting conservative Dirichlet diffusion matrix is symmetric positive definite. Quantify why the arithmetic mean can be seriously wrong when a_+/a_- is large.

Problem 6. [20 points; choose Problem 6 or Problem 7] Let $g \in C^3[0,1]$ and

$$I(\lambda) = \int_0^1 e^{i\lambda(x-x^2/2)} g(x)\, dx, \qquad \lambda \to +\infty.$$

Find the expansion through an absolute remainder $O(\lambda^{-3/2})$. You may use

$$\int_0^\infty e^{-is^2/2}\, ds = e^{-i\pi/4}\sqrt{\frac{\pi}{2}}.$$

Problem 7. [20 points; choose Problem 6 or Problem 7] Let $D = \operatorname{diag}(d_1, \dots, d_n)$ with $d_i > 0$, let $b \neq 0$, and let $R > 0$. Determine the unique solution of

$$\min_{\|x\|_2 \le R} \left(\frac{1}{2} x^T D x - b^T x \right).$$

Give a complete characterization of the interior and boundary cases and a reliable algorithm for the boundary case, with justification. Finally determine the limit of $x^\star/R$ as $R \downarrow 0$.