Mock Qualifying Examination 04

Computational and Applied Mathematics

Total: 100 points

Instructions. Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

Problem 1. [10 points] Suppose

$$f(x) = (x-\alpha)^m h(x), \qquad m \geq 2, \qquad h \in C^3, \qquad h(\alpha) \neq 0.$$

For $x_k \neq \alpha$, consider

$$x_{k+1} = x_k - \frac{f(x_k) f'(x_k)}{f'(x_k)^2 - f(x_k) f''(x_k)}.$$

Show that the iteration is well defined for all sufficiently close nonzero errors. Prove that its convergence is at least quadratic, determine the coefficient of the quadratic error term, and state the order when that coefficient is nonzero. An iterate equal to α is regarded as terminated.

Problem 2. [20 points] Let w be continuous and strictly positive on $[a, b]$, and let $n \geq 1$. Determine whether there is a unique quadrature rule of the form

$$\int_a^b w(x) f(x)\, dx = \lambda_0 f(a) + \sum_{j=1}^{n} \lambda_j f(x_j), \qquad a < x_1 < \cdots < x_n < b,$$

that is exact for every polynomial of degree at most $2n$. If so, characterize its nodes and prove that every weight is positive. Your argument should also establish uniqueness.

Problem 3. [20 points] Given $A, B \in \mathbb{R}^{m \times n}$, solve

$$\min_{Q \in \mathbb{R}^{n \times n},\, Q^T Q = I} \|AQ - B\|_F\,.$$

Give the minimum value and characterize all minimizers. In particular, give a necessary and sufficient condition for the minimizer to be unique.

Problem 4. [15 points] Consider a one-step finite-difference formula on the real line,

$$U_j^{k+1} = \sum_{\ell=-p}^{q} a_\ell U_{j+\ell}^k, \qquad p, q \geq 0,$$

intended to approximate $u_t + c u_x = 0$. Suppose $h, \tau \to 0$ with the Courant number $\nu = c\tau/h$ fixed, and suppose the formula converges at each fixed time for every smooth compactly supported initial function. Prove the necessary condition

$$-q \leq \nu \leq p.$$

No sign assumption is made on the coefficients a_ℓ.

Problem 5. [15 points] Let $u \in C^6[0, 1]$ solve

$$-u'' = f, \qquad u'(0) = 0, \qquad u(1) = \beta.$$

On $x_j = jh$, $h = 1/N$, consider

$$\frac{2U_j - U_{j-1} - U_{j+1}}{h^2} = f(x_j) \quad (1 \leq j < N), \qquad \frac{2(U_0 - U_1)}{h^2} = f(0), \qquad U_N = \beta.$$

Determine the local truncation order at the boundary point $x = 0$ and the global convergence order in the maximum norm. Prove both conclusions.

Problem 6. [20 points; choose Problem 6 or Problem 7] On $\Omega = (0, \pi)^2$, consider

$$-\Delta u + \varepsilon \cos x \cos y \, u = \lambda(\varepsilon) u, \qquad u|_{\partial\Omega} = 0, \qquad \|u\|_{L^2(\Omega)} = 1.$$

The unperturbed eigenvalue 5 is double. Assuming formal expansions of the eigenpairs in powers of ε, determine the first-order terms of the two eigenvalues issuing from 5 and their leading normalized eigenfunctions. Explain why treating either unperturbed coordinate eigenfunction as though the eigenvalue were simple gives an incomplete solvability condition.

Problem 7. [20 points; choose Problem 6 or Problem 7] Let a_1, a_2, a_3 be the vertices of a nondegenerate triangle. Determine the unique minimizer of

$$F(x) = \sum_{i=1}^{3} \|x - a_i\|_2 \,, \qquad x \in \mathbb{R}^2,$$

in the following sense: give a geometric characterization that covers both the case in which an angle of the triangle is at least 120° and the case in which all three angles are smaller. Prove existence, uniqueness, and your characterization without assuming in advance that the minimizer lies inside the triangle.