Mock Qualifying Examination 03

Computational and Applied Mathematics

Total: 100 points

Instructions. Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

Problem 1. [10 points] Let $f \in C^1(\mathbb{R})$ have a zero α and satisfy

$$0 < m \le f'(x) \le M \qquad (x \in \mathbb{R}).$$

For the iteration $x_{k+1} = x_k - \omega f(x_k)$ with a constant ω, determine all positive values of ω that guarantee convergence to α from every starting point for every such f. Among them, find the ω giving the smallest worst-case contraction factor, and give that factor. Justify the sharp endpoints of your answer.

Problem 2. [20 points] Let $f \in C^2[a,b]$ satisfy $f'' > 0$. Among all affine functions p, determine the unique minimizer of

$$\max_{a \le x \le b} |f(x) - p(x)|$$

and the value of the minimum. Prove both optimality and uniqueness without quoting a general alternation theorem. Apply your result to $f(x) = e^x$ on $[-1, 1]$.

Problem 3. [20 points] Let $A = A^T \in \mathbb{R}^{n\times n}$ have a simple smallest eigenvalue $\lambda_1 < \lambda_2 \le \cdots \le \lambda_n$, with unit eigenvector v_1. For a unit vector x_k close to v_1, define

$$\rho_k = x_k^T A x_k, \qquad (A - \rho_k I) y_k = x_k, \qquad x_{k+1} = \frac{y_k}{\|y_k\|_2}.$$

If $x_k = v_1$, regard the iteration as terminated. Prove that the iteration is otherwise well defined in a sufficiently small neighborhood of v_1 and that

$$\tan \angle(x_{k+1}, v_1) \le C \tan^3 \angle(x_k, v_1)$$

there. Determine the corresponding local order of convergence of the Rayleigh quotients ρ_k to λ_1.

Problem 4. [15 points] Let $L = L^T \preceq 0$ be a periodic finite-difference diffusion matrix and let $Q = Q^T \preceq 0$ be a diagonal reaction matrix. They need not commute. For $U' = (L+Q)U$, consider

$$S_\tau = e^{\tau Q/2} e^{\tau L} e^{\tau Q/2}.$$

By direct expansion, determine the first nonzero term in $S_\tau - e^{\tau(L+Q)}$ and, for a fixed spatial grid, infer the temporal order. Prove an ℓ^2 stability statement valid for every $\tau > 0$, and compare the order with the nonsymmetric product $e^{\tau Q} e^{\tau L}$. What changes when $LQ = QL$?

Problem 5. [15 points] On a periodic interval of fixed length with N grid points and $h = L/N$, let A_h be the standard matrix for $-u_{xx}$,

$$(A_h U)_j = \frac{-U_{j-1} + 2U_j - U_{j+1}}{h^2},$$

and consider weighted Jacobi iteration for $A_h U = F$, where F has zero mean,

$$U^{(k+1)} = U^{(k)} + \omega D^{-1}(F - A_h U^{(k)}), \qquad D = \frac{2}{h^2} I.$$

Choose $\omega > 0$ to minimize the largest one-step amplification over the Fourier-symbol band $\pi/2 \leq |\theta| \leq \pi$, and determine the resulting factor. Explain why this good high-frequency behavior does not make weighted Jacobi a mesh-independent solver for the periodic Poisson equation when the iteration is restricted to mean-zero vectors.

Problem 6. [20 points; choose Problem 6 or Problem 7] Let $0 < \varepsilon \ll 1$, $\zeta > 0$, $F > 0$, and $\gamma, \sigma \in \mathbb{R}$. Consider

$$y'' + y = \varepsilon \left[-2\zeta y' - \gamma y^3 + F \cos\big((1 + \varepsilon\sigma)t\big) \right].$$

Obtain the leading slow modulation equations on $t = O(\varepsilon^{-1})$. For a response locked to the forcing frequency, derive an equation relating its steady physical amplitude to F, ζ, γ, and σ. Explain why more than one steady amplitude can occur, but do not analyze branch stability.

Problem 7. [20 points; choose Problem 6 or Problem 7] Consider the linear program

$$\min c^T x \qquad \text{subject to} \qquad Ax = b, \quad x \geq 0,$$

where $A \in \mathbb{R}^{m \times n}$ has rank $m < n$. Let B be a set of m column indices such that A_B is nonsingular, and suppose the associated basic feasible solution is nondegenerate:

$$x_B^\star = A_B^{-1} b > 0, \qquad x_N^\star = 0.$$

Set $y = A_B^{-T} c_B$. Prove that $x^\star$ is the unique optimal solution if and only if

$$c_N - A_N^T y > 0$$

componentwise. Your proof must explain what a negative or a zero reduced cost implies and where nondegeneracy is used.