# Mock Qualifying Examination 02

Computational and Applied Mathematics

**Total: 100 points**

**Instructions.** Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

**Problem 1. [10 points]** Let $\phi \in C^1([a,b])$ map $[a,b]$ into itself and be decreasing. Suppose that, for some $0 < q < 1$,

$$\bigl|\phi'(x)\phi'(\phi(x))\bigr| \le q, \qquad a \le x \le b.$$

Prove that, for every $x_0 \in [a,b]$, the iteration $x_{k+1} = \phi(x_k)$ converges to a unique fixed point $\alpha$. Give a two-step error estimate that is valid from the first iterate onward.

**Problem 2. [20 points]** For $f \in C[0,1]$, define

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \le x \le 1.$$

Prove that $B_n f \to f$ uniformly on $[0,1]$. If $f \in C^2[0,1]$, determine the uniform limit on $[0,1]$ of

$$n\bigl(B_n f - f\bigr).$$

The endpoint cases and uniformity of the remainder are part of the problem.

**Problem 3. [20 points]** Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite and let $C \in \mathbb{R}^{n\times n}$ be symmetric. Consider

$$AX + XA = C.$$

Prove that the equation has a unique solution, and give an algorithm with $O(n^3)$ arithmetic cost. For the linear operator $\mathcal{L}(X) = AX + XA$, determine its condition number induced by the Frobenius norm. Finally, decide whether $C \succeq 0$ implies $X \succeq 0$, and justify your answer without assuming that $A$ and $C$ commute.

**Problem 4. [15 points]** For the periodic advection equation $u_t + au_x = 0$, where $a \ne 0$, consider

$$\frac{U_j^{m+1} - U_j^{m-1}}{2\tau} + a\frac{U_{j+1}^m - U_{j-1}^m}{2h} = 0.$$

Determine its consistency order and the exact range of $a\tau/h$ for which it is stable uniformly over periodic grids for arbitrary two-level starting data. Explain both the failure at the stability endpoint and the role of the second mode carried by this three-level scheme.

**Problem 5. [15 points]** Let $\varepsilon > 0$ and $b \ne 0$ be constants. On the uniform grid $x_j = jh$, $0 \le j \le N$, with $h = 1/N$, apply centered differences to

$$-\varepsilon u'' + bu' = f, \qquad u(0) = \alpha, \qquad u(1) = \beta.$$

Determine precisely when the resulting discrete boundary-value problem satisfies the following comparison principle: if $L_h V \ge L_h W$ at every interior node and $V_0 \ge W_0$, $V_N \ge W_N$, then $V_j \ge W_j$ for every $j$, where $L_h$ is the centered discrete operator above. When this condition fails, analyze the homogeneous difference equation and show how the failure appears as a mesh-scale oscillation. Include the limiting cases in which one off-diagonal coefficient vanishes.

**Problem 6. [20 points; choose Problem 6 or Problem 7]** Consider

$$u_t = (u^2)_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0,$$

and prescribe a total mass $M > 0$. Construct a nonnegative compactly supported solution whose shape changes only by rescaling. Determine its profile, peak decay, and support growth, with the constant fixed by $M$. Verify the equation where the solution is positive and check the flux at each moving interface. No weak-solution theory is required.

**Problem 7. [20 points; choose Problem 6 or Problem 7]** For the scalar controlled system

$$x_{k+1} = ax_k + u_k,$$

minimize, over $u_0, \dots, u_{N-1}$,

$$\sum_{k=0}^{N-1} \left(qx_k^2 + ru_k^2\right) + q_f x_N^2, \qquad q, r > 0, \qquad q_f \geq 0.$$

Determine the optimal feedback controls and the value function for every finite horizon. As the remaining horizon tends to infinity, prove that the coefficient in the value function converges to a unique positive limit, independently of $q_f$, and prove that the corresponding limiting closed-loop system is asymptotically stable.