Mock Qualifying Examination 01

Computational and Applied Mathematics

Total: 100 points

Instructions. Problems 1–5 are required. Choose **one** of Problems 6 and 7. If both are attempted, only the higher score will be counted. Give complete arguments and state any additional assumptions you use.

Problem 1. [10 points] Let $x_{k+1} = \phi(x_k)$ converge to a fixed point α, where $\phi \in C^3$ near α and

$$q := \phi'(\alpha), \qquad 0 < |q| < 1.$$

Assume that $x_k \neq \alpha$ for every k. Whenever the denominator below is nonzero, define

$$y_k = x_k - \frac{(x_{k+1} - x_k)^2}{x_{k+2} - 2x_{k+1} + x_k}.$$

Show that the denominator is nonzero for all sufficiently large k, and find

$$\lim_{k\to\infty} \frac{y_k - \alpha}{(x_k - \alpha)^2}$$

in terms of derivatives of ϕ at α.

Problem 2. [20 points] Let P and Q be real polynomials of degree at most two, normalized by $Q(0) = 1$. Determine all such pairs for which

$$Q(z)e^z - P(z) = O(z^5), \qquad z \to 0.$$

For the resulting rational function $R = P/Q$:

(a) determine whether any pole lies in the closed left half-plane;

(b) decide whether $|R(z)| \leq 1$ for every $\operatorname{Re} z \leq 0$;

(c) determine the limit of $R(z)$ along the negative real axis as $z \to -\infty$, and explain what this says about the decay of very stiff modes when R is used as a stability function.

Problem 3. [20 points] Let $J_n \in \mathbb{R}^{n\times n}$ be defined by

$$J_n e_1 = 0, \qquad J_n e_j = e_{j-1} \quad (2 \leq j \leq n),$$

and let $A = I + MJ_n$, where $M > 0$. Apply GMRES in exact arithmetic to $Ax = b$ with $b = e_n$ and $x_0 = 0$.

For $0 \leq k < n$, determine the exact value of

$$\frac{\|r_k\|_2}{\|r_0\|_2},$$

where r_k is the residual after k GMRES steps. At which step must GMRES terminate? Explain why the spectrum of A alone gives a misleading prediction for this example when M is large.

Problem 4. [15 points] On a periodic rectangular grid, let D_{xx} and D_{yy} denote the standard centered second-difference matrices in the two coordinate directions. Consider the update for the heat equation $u_t = u_{xx} + u_{yy}$:

$$\left(I - \frac{\tau}{2}D_{xx}\right)\left(I - \frac{\tau}{2}D_{yy}\right)U^{m+1} = \left(I + \frac{\tau}{2}D_{xx}\right)\left(I + \frac{\tau}{2}D_{yy}\right)U^m.$$

Determine its order of consistency in time and space, establish an ℓ^2 stability statement valid for all $\tau, h_x, h_y > 0$, and describe an implementation whose cost is linear in the number of grid points per time step.

Problem 5. [15 points] Let $A \in \mathbb{R}^{s\times s}$ be symmetric and nonzero, and consider the periodic system

$$U_t + AU_x = 0.$$

Define $|A| = (A^2)^{1/2}$, the symmetric positive semidefinite square root of A^2, and consider

$$U_j^{m+1} = U_j^m - \frac{\nu}{2}A(U_{j+1}^m - U_{j-1}^m) + \frac{\nu}{2}|A|(U_{j+1}^m - 2U_j^m + U_{j-1}^m), \qquad \nu = \frac{\tau}{h}.$$

Find a necessary and sufficient condition for ℓ^2 stability. Under that condition, explain how the update chooses the upwind direction for each wave family, determine the leading numerical diffusion in each family, and describe what happens at the stability endpoint to every wave family whose speed has maximal magnitude.

Problem 6. [20 points; choose Problem 6 or Problem 7] For fixed $a \in (-1, 1)$, consider

$$I(\lambda; a) = \int_{-1}^{1} \exp\left\{\lambda\left(\frac{a}{2}x^2 - \frac{1}{4}x^4\right)\right\} dx, \qquad \lambda \to \infty.$$

Find the leading term for each of the regimes $a < 0$, $a = 0$, and $a > 0$. Explain why these three fixed-a formulas cannot be uniform as $a \to 0$, and identify the scale of a on which the transition occurs.

Problem 7. [20 points; choose Problem 6 or Problem 7] Let $H \in \mathbb{R}^{n\times n}$ be symmetric positive definite, let $A \in \mathbb{R}^{m\times n}$ have full row rank with $m < n$, and suppose $Ax = b$ is feasible. Let $x^\star$ be the solution of

$$\min_{Ax=b}\left(\frac{1}{2}x^T Hx - c^T x\right),$$

and let $\lambda^\star$ be its Lagrange multiplier under the sign convention

$$Hx^\star - c + A^T\lambda^\star = 0.$$

For $\mu > 0$, define

$$x_\mu = \arg\min_x\left\{\frac{1}{2}x^T Hx - c^T x + \frac{\mu}{2}\|Ax - b\|_2^2\right\}, \qquad \lambda_\mu = \mu(Ax_\mu - b).$$

Prove quantitative convergence of x_μ and λ_μ to $x^\star$ and $\lambda^\star$ as $\mu \to \infty$. Then determine the asymptotic order, in μ, of $\kappa_2(H + \mu A^T A)$. Interpret the two conclusions together.