Homework Assignment 2

2025.11.14

Problem 1. (15 points)

(a) Show that a nonpolyhedral closed convex cone need not be retractive, by using as an example the cone $C = \{(u, v, w) \mid \|(u, v)\| \leq w\}$, the recession direction $d = (1, 0, 1)$, and the corresponding asymptotic sequence $\left\{\left(k, \sqrt{k}, \sqrt{k^2 + k}\right)\right\}$.
(b) Verify that the cone C of part (a) can be written as the intersection of an infinite number of closed halfspaces, thereby showing that a nested set sequence obtained by intersection of an infinite number of retractive nested set sequences need not be retractive.

Solution 1 *(a) First, we show that $\left\{x_k = (k, \sqrt{k}, \sqrt{k^2 + k})\right\}$ is an asymptotic sequence for the cone C in the direction $d = (1, 0, 1)$.*

We have

$$\|x_k\| = \sqrt{k^2 + k + k^2 + k} = \sqrt{2k^2 + 2k} = \sqrt{2}\sqrt{k^2 + k}$$

Thus,

$$\frac{x_k}{\|x_k\|} = \left(\frac{k}{\sqrt{2}\sqrt{k^2 + k}}, \frac{\sqrt{k}}{\sqrt{2}\sqrt{k^2 + k}}, \frac{\sqrt{k^2 + k}}{\sqrt{2}\sqrt{k^2 + k}}\right) = \left(\frac{k}{\sqrt{2}\sqrt{k^2 + k}}, \frac{\sqrt{k}}{\sqrt{2}\sqrt{k^2 + k}}, \frac{1}{\sqrt{2}}\right)$$

As $k \to \infty$, we have

$$\lim_{k \to \infty} \frac{x_k}{\|x_k\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(1, 0, 1) = \frac{d}{\|d\|}$$

Next, we check that $x_k - d \in C$ not true for all k sufficiently large. Which equivalently means checking if $\left\|(k - 1, \sqrt{k}, \sqrt{k^2 + k} - 1)\right\| \leq \sqrt{k^2 + k} - 1 \Longleftrightarrow -2k \leq -2\sqrt{k^2 + k}$ which is not true for all k sufficiently large.

(b) We want to show $C = \bigcap_{\theta \in [0, 2\pi]} H_\theta$, where $H_\theta = \{(u, v, w) \mid u \cos\theta + v \sin\theta - w \leq 0\}$ are closed halfspaces. This is true since the inequality $\|(u, v)\| \leq w$ is equivalent to

$$u \cos\theta + v \sin\theta \leq w \quad \textit{for all } \theta \in [0, 2\pi]$$

Each H_θ is a closed halfspace defined by the normal $(\cos\theta, \sin\theta, -1)$. Each H_θ is a polyhedral set and therefore retractive.

Problem 2. (15 points)

Let C_1 and C_2 be nonempty convex subsets of $\mathbf{R}^n$, and let B denote the unit ball in $\mathbf{R}^n$, $B = \{x \mid \|x\| \le 1\}$. A hyperplane H is said to separate strongly C_1 and C_2 if there exists an $\epsilon > 0$ such that $C_1 + \epsilon B$ is contained in one of the open halfspaces associated with H and $C_2 + \epsilon B$ is contained in the other. Show the following three conditions are equivalent.

(i) There exists a hyperplane separating strongly C_1 and C_2.

(ii) There exists a vector $\alpha \in \mathbf{R}^n$ such that $\inf_{x \in C_1} \alpha' x > \sup_{x \in C_2} \alpha' x$.

(iii) $\inf_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\| > 0$, i.e., $0 \notin \mathrm{cl}\,(C_2 - C_1)$.

Solution 2 *(i) $\implies$ (ii): There exists a hyperplane $H = \{x \mid \alpha' x = b\}$ and an $\epsilon > 0$ such that $C_1 + \epsilon B$ and $C_2 + \epsilon B$ are in opposite open halfspaces. Assume $C_1 + \epsilon B \subset \{x \mid \alpha' x > b\}$ and $C_2 + \epsilon B \subset \{x \mid \alpha' x < b\}$.*

For any $x_1 \in C_1$ and $z \in B$, we have $\alpha'(x_1 + \epsilon z) > b$, which means $\alpha' x_1 + \inf_{z \in B}(\epsilon \alpha' z) > b$ $\inf_{z \in B}(\alpha' z) = -\sup_{z \in B}(-\alpha' z) = -\|\alpha\|$. So, $\alpha' x_1 - \epsilon \|\alpha\| > b$ for all $x_1 \in C_1$. Taking the infimum over x_1:

$$\inf_{x_1 \in C_1} \alpha' x_1 \ge b + \epsilon \|\alpha\|$$

Similarly,

$$\sup_{x_2 \in C_2} \alpha' x_2 \le b - \epsilon \|\alpha\|$$

Combining these, we have:

$$\sup_{x_2 \in C_2} \alpha' x_2 \le b - \epsilon \|\alpha\| < b + \epsilon \|\alpha\| \le \inf_{x_1 \in C_1} \alpha' x_1$$

(ii) $\implies$ (iii): There exists $\alpha \neq 0$ such that $\inf_{x_1 \in C_1} \alpha' x_1 > \sup_{x_2 \in C_2} \alpha' x_2$. Let $\gamma_1 = \inf_{x_1 \in C_1} \alpha' x_1$ and $\gamma_2 = \sup_{x_2 \in C_2} \alpha' x_2$. Let $\delta = \gamma_1 - \gamma_2 > 0$. Then $\|\alpha\| \|x_1 - x_2\| \ge \alpha'(x_1 - x_2) = \alpha' x_1 - \alpha' x_2 \ge \gamma_1 - \gamma_2 = \delta$. This implies $\|x_1 - x_2\| \ge \frac{\delta}{\|\alpha\|}$ for all $x_1 \in C_1, x_2 \in C_2$. Therefore, $\inf_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\| \ge \frac{\delta}{\|\alpha\|} > 0$.

(iii) $\implies$ (i): Let $\delta = \inf_{x_1 \in C_1, x_2 \in C_2} \|x_1 - x_2\| > 0$. Let $D = C_1 - C_2 = \{x_1 - x_2 \mid x_1 \in C_1, x_2 \in C_2\}$ is convex since C_1 and C_2 are convex. We have $\inf_{d \in D} \|d\| = \delta > 0$. This implies that the convex set D is disjoint from the open ball $B(0, \delta) = \{x \mid \|x\| < \delta\}$. By the Separating Hyperplane Theorem, there exists a hyperplane $H = \{x \mid \alpha' x = b\}$ that separates D and $B(0, \delta)$. Let $\epsilon = \frac{\delta}{2\|\alpha\|} > 0$. Then, for any $x_1 \in C_1$ and $z \in B$, we have:

$$\alpha'(x_1 + \epsilon z - x_2) = \alpha'(x_1 - x_2) + \epsilon \alpha' z \ge b + \epsilon(-\|\alpha\|) = b - \frac{\delta}{2}$$

Similarly, for any $x_2 \in C_2$ and $z \in B$, we have:

$$\alpha'(x_1 - (x_2 + \epsilon z)) = \alpha'(x_1 - x_2) - \epsilon \alpha' z \le b + \epsilon \|\alpha\| = b + \frac{\delta}{2}$$

Thus, $C_1 + \epsilon B$ and $C_2 + \epsilon B$ are contained in opposite open halfspaces defined by H.

Problem 3. (15 points)

Let $F : \mathbf{R}^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function of two vectors $x \in \mathbf{R}^n$ and $z \in \mathbf{R}^m$, and let

$$X = \left\{ x \mid \inf_{z \in \mathbf{R}^m} F(x, z) < \infty \right\}.$$

Assume that the function $F(x, \cdot)$ is closed for each $x \in X$. Show that:
(a) If for some $\bar{x} \in X$, the minimum of $F(\bar{x}, \cdot)$ over $\mathbf{R}^m$ is attained at a nonempty and compact set, the same is true for all $x \in X$.
(b) If the functions $F(x, \cdot)$ are differentiable for all $x \in X$, they have the same asymptotic slopes along all directions, i.e., for each $d \in \mathbf{R}^m$, the value of $\lim_{\alpha \to \infty} \nabla_z F(x, z + \alpha d)'d$ is the same for all $x \in X$ and $z \in \mathbf{R}^m$.

Solution 3 *(a) Let $g_x(z) = F(x, z)$. We are given that for a specific $\tilde{x} \in X$, $g_{\tilde{x}}(z)$ is a closed convex function and its set of minimizers $M(\tilde{x}) = \arg\min_z g_{\tilde{x}}(z)$ is nonempty and compact.*

For a closed proper convex function g, the set of minimizers is nonempty and compact if and only if g recession function $r_g(d)$ satisfies $r_g(d) > 0$ for all $d \neq 0$. The recession function of $g_x(z) = F(x, z)$ is

$$r_{g_x}(d_z) = \lim_{\alpha \to \infty} \frac{F(x, z_0 + \alpha d_z) - F(x, z_0)}{\alpha}$$

for any z_0 in the domain of g_x.

So we have $r_F(0, d_z) = r_{g_{\tilde{x}}}(d_z) > 0$ for all $d_z \neq 0$. Since $x \in X$, we know $\inf_z F(x, z) < \infty$. Thus $M(x) = \arg\min_z F(x, z)$ is nonempty. The set of minimizers $M(x)$ is a level set of g_x, and since g_x is closed, $M(x)$ is closed. A closed and bounded set in R^m is compact. Thus, for all $x \in X$, $M(x)$ is nonempty and compact.

(b) Let $g_x(z) = F(x, z)$. We have

$$\lim_{\alpha \to \infty} \nabla_z F(x, z + \alpha d)'d = r_{g_x}(d)$$

This is because $\nabla g_x(z+\alpha d)'d$ is non-decreasing in α and its limit is $\sup_\alpha \nabla g_x(z+\alpha d)'d$, which equals $r_{g_x}(d)$. As we established in part (a), $r_{g_x}(d) = r_F(0, d)$. This value $r_F(0, d)$ depends only on F and d, and is independent of x and z. Therefore, the value of the asymptotic slope is the same for all $x \in X$ and $z \in R^m$.

Problem 4. (15 points)

Let $f : \mathbf{R}^n \mapsto \mathbf{R}$ be the function

$$f(x) = \frac{1}{p} \sum_{i=1}^{n} |x_i|^p$$

where $1 < p$. Show that the conjugate is

$$f^\star(y) = \frac{1}{q} \sum_{i=1}^{n} |y_i|^q ,$$

where q is defined by the relation

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Solution 4

$$\begin{aligned} f^*(y) = \sup_{x \in \mathbb{R}^n} \{y'x - f(x)\} &= \sup_{x \in \mathbb{R}^n} \left\{ \sum_{i=1}^n y_i x_i - \frac{1}{p} \sum_{i=1}^n |x_i|^p \right\} \\ &= \sum_{i=1}^n \sup_{x_i \in \mathbb{R}} \left\{ y_i x_i - \frac{1}{p} |x_i|^p \right\} \\ (x_i^* = |y_i|^{q-1}) &= \sum_{i=1}^n \left(1 - \frac{1}{p}\right) |y_i|^q \\ &= \frac{1}{q} \sum_{i=1}^n |y_i|^q \end{aligned}$$

Problem 5. (15 points)

(a) Show that if $f_1 : \mathbf{R}^n \mapsto (-\infty, \infty]$ and $f_2 : \mathbf{R}^n \mapsto (-\infty, \infty]$ are closed proper convex functions, with conjugates denoted by ${f_1}^\star$ and ${f_2}^\star$, respectively, we have

$$f_1(x) \le f_2(x), \quad \forall x \in \mathbf{R}^n$$

if and only if

$$f_1^\star(y) \ge f_2^\star(y), \quad \forall y \in \mathbf{R}^n.$$

(b) Show that if C_1 and C_2 are nonempty closed convex sets, we have

$$C_1 \subset C_2$$

if and only if

$$\sigma_{C_1}(y) \le \sigma_{C_2}(y), \quad \forall y \in \mathbf{R}^n.$$

Construct an example showing that closedness of C_1 and C_2 is a necessary assumption.

Solution 5 *(a) "$\Longrightarrow$":*

$$f_1^*(y) = \sup_x \{y'x - f_1(x)\} \ge \sup_x \{y'x - f_2(x)\} = f_2^*(y)$$

"$\Longleftarrow$": Since f_1 and f_2 are closed proper convex functions, $f = f^{**}$.

$$f_1(x) = f_1^{**}(x) = \sup_y \{x'y - f_1^*(y)\} \le \sup_y \{x'y - f_2^*(y)\} = f_2^{**}(x) = f_2(x)$$

(b) Let $\delta_{C_1}(x)$ and $\delta_{C_2}(x)$ be the indicator functions for the sets C_1 and C_2.

$$C_1 \subset C_2 \iff \delta_{C_2}(x) \le \delta_{C_1}(x), \forall x \iff \delta_{C_2}^*(y) \ge \delta_{C_1}^*(y), \forall y \iff \sigma_{C_2}(y) \ge \sigma_{C_1}(y) \quad \forall y$$

Counterexample: *Let* $C_1 = [0, 1]$, $C_2 = (0, 1)$.

$$\sigma_{C_1}(y) = \sup_{x \in [0,1]} \{yx\} = \begin{cases} y & \text{if } y > 0 \\ 0 & \text{if } y \leq 0 \end{cases} = \max(0, y)$$

$$\sigma_{C_2}(y) = \sup_{x \in (0,1)} \{yx\} = \begin{cases} y & \text{if } y > 0 \\ 0 & \text{if } y \leq 0 \end{cases} = \max(0, y)$$

Here, $\sigma_{C_1}(y) = \sigma_{C_2}(y)$, *so* $\sigma_{C_1}(y) \leq \sigma_{C_2}(y)$ *is satisfied. However,* $C_1 \not\subset C_2$.

Problem 6. (15 points)

Consider a function ϕ of two real variables x and z taking values in compact intervals X and Z, respectively. Assume that for each $z \in Z$, the function $\phi(\cdot, z)$ is minimized over X at a unique point denoted $\hat{x}(z)$. Similarly, assume that for each $x \in X$, the function $\phi(x, \cdot)$ is maximized over Z at a unique point denoted $\hat{z}(x)$. Assume further that the functions $\hat{x}(z)$ and $\hat{z}(x)$ are continuous over Z and X, respectively. Show that ϕ has a saddle point (x^*, z^*). Use this to investigate the existence of saddle points of $\phi(x, z) = x^2 + z^2$ over $X = [0, 1]$ and $Z = [0, 1]$.

Solution 6

$$\phi(x^*, z) \leq \phi(x^*, z^*) \leq \phi(x, z^*)$$

which gives:

$$x^* = \hat{x}(z^*)$$

$$z^* = \hat{z}(x^*)$$

Define $T : X \times Z \to X \times Z$ *as*

$$T(x, z) = (\hat{x}(z), \hat{z}(x))$$

Since X *and* Z *are compact intervals, they are compact and convex,* $X \times Z$ *is also compact, convex.Since* $\hat{x}(z)$ *and* $\hat{z}(x)$ *are continuous functions,* $T(x, z)$ *is a continuous function from* $X \times Z$ *to* $X \times Z$. *By the Brouwer Fixed-Point Theorem, there exists a point* $(x^*, z^*) \in X \times Z$ *such that* $T(x^*, z^*) = (x^*, z^*)$.

For ϕ, *we have* $(x^*, z^*) = (0, 1)$ *since:*

$$\phi(0, z) \leq \phi(0, 1) \leq \phi(x, 1)$$

$$\iff 0^2 + z^2 \leq 0^2 + 1^2 \leq x^2 + 1^2$$

$$\iff z^2 \leq 1 \leq x^2 + 1$$

Problem 7. (10 points)

let $F(x,u) = f_1(x) + f_2(Ax+u)$, where A is an $m \times n$ matrix, and $f_1 : \mathbf{R}^n \mapsto (-\infty, \infty]$ and $f_2 : \mathbf{R}^m \mapsto (-\infty, \infty]$ are closed convex functions. Show that the dual function is

$$q(\mu) = -f_1^\star(A'\mu) - f_2^\star(-\mu)$$

where $f_1^\star$ and $f_2^\star$ are the conjugate functions of f_1 and f_2, respectively.

Solution 7 *Let* $p = \inf_x F(x,0) = \inf_x\{f_1(x) + f_2(Ax)\}$, $L(x,u,\mu) = F(x,u) + \mu' u$*:*

$$\begin{aligned} q(\mu) = \inf_{x\in\mathbb{R}^n, u\in\mathbb{R}^m}\{L(x,u,\mu)\} &= \inf_{x,u}\{f_1(x) + f_2(Ax+u) + \mu'u\} \\ &= \inf_{x\in\mathbb{R}^n}\left\{f_1(x) + \inf_{u\in\mathbb{R}^m}\{f_2(Ax+u) + \mu'u\}\right\} \end{aligned}$$

Set $z = Ax + u$*. Then* $u = z - Ax$*.*

$$\begin{aligned} \inf_u\{f_2(Ax+u) + \mu'u\} &= \inf_z\{f_2(z) + \mu'(z - Ax)\} \\ &= \inf_z\{f_2(z) + \mu'z - \mu'Ax\} \\ &= -f_2^*(-\mu) - \mu'Ax \end{aligned}$$

So the dual function is:

$$\begin{aligned} q(\mu) &= \inf_{x\in\mathbb{R}^n}\{f_1(x) + (-\mu'Ax - f_2^*(-\mu))\} \\ &= \inf_{x\in\mathbb{R}^n}\{f_1(x) - \mu'Ax\} - f_2^*(-\mu) \\ &= \inf_{x\in\mathbb{R}^n}\{f_1(x) - (A'\mu)'x\} - f_2^*(-\mu) \\ &= -f_1^*(A'\mu) - f_2^*(-\mu) \end{aligned}$$