Homework Assignment 3

Problem 1. (15 points)

Consider the class of problems

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in X, \quad g_j(x) \leq u_j, \quad j = 1, \ldots, r, \end{array}$$

where $u = (u_1, \ldots, u_r)$ is a vector parameterizing the right-hand side of the constraints. Given two distinct values $\bar{u}$ and $\tilde{u}$ of u, let $\bar{f}$ and $\tilde{f}$ be the corresponding optimal values, and assume that $\bar{f}$ and $\tilde{f}$ are finite. Assume further that $\bar{\mu}$ and $\tilde{\mu}$ are corresponding dual optimal solutions and that there is no duality gap. Show that

$$\tilde{\mu}'(\tilde{u} - \bar{u}) \leq \bar{f} - \tilde{f} \leq \bar{\mu}'(\tilde{u} - \bar{u}).$$

Solution 1

$$D(\mu) = \inf_{x \in X} \{f(x) + \mu' g(x)\}.$$

$$q(\mu, u) = \inf_{x \in X} \{f(x) + \mu'(g(x) - u)\} = D(\mu) - \mu' u.$$

$$\bar{f} = p(\bar{u}) = \sup_{\mu \geq 0} (D(\mu) - \mu' \bar{u}) = D(\bar{\mu}) - \bar{\mu}' \bar{u}$$

$$\tilde{f} = p(\tilde{u}) = \sup_{\mu \geq 0} (D(\mu) - \mu' \tilde{u}) = D(\tilde{\mu}) - \tilde{\mu}' \tilde{u}$$

where $\bar{\mu}$ and $\tilde{\mu}$ is the corresponding dual optimal solutions. We have:

$$\bar{f} = D(\bar{\mu}) - \bar{\mu}' \bar{u} \geq D(\tilde{\mu}) - \tilde{\mu}' \bar{u} = (\tilde{f} + \tilde{\mu}' \tilde{u}) - \tilde{\mu}' \bar{u} = \tilde{f} + \tilde{\mu}'(\tilde{u} - \bar{u}).$$

$$\implies \tilde{\mu}'(\tilde{u} - \bar{u}) \leq \bar{f} - \tilde{f}.$$

Otherside is similar.

Problem 2. (15 points)

Let $g_j : R^n \mapsto R, j = 1, \ldots, r$, be convex functions over the nonempty convex subset of R^n. Show that the system

$$g_j(x) < 0, \quad j = 1, \ldots, r,$$

has no solution within X if and only if there exists a vector $\mu \in R^r$ such that

$$\sum_{j=1}^{r} \mu_j = 1, \quad \mu \geq 0,$$

$$\mu' g(x) \geq 0, \quad \forall x \in X.$$

Hint: Consider the convex program

$$\begin{aligned} &\underset{x,y}{\text{minimize}} \; y \\ &\text{subject to} \;\; x \in X, y \in R, g_j(x) \leq y, \quad j = 1, \ldots, r, \end{aligned} \tag{1}$$

Solution 2 *Consider the convex program*

$$\begin{aligned} &\underset{x,y}{\text{minimize}} \; y \\ &\textit{subject to} \;\; x \in X, y \in R, g_j(x) \leq y, \quad j = 1, \ldots, r, \end{aligned} \tag{2}$$

Let v^* be the optimal value of this problem. The system $g_j(x) < 0$ for $j = 1, \ldots, r$ has no solution in X if and only if for all $x \in X$, $\max_j g_j(x) \geq 0$. This implies that the optimal value of the auxiliary problem satisfies $v^* \geq 0$.

Now, consider the Lagrangian of the auxiliary problem:

$$L(x, y, \lambda) = y + \sum_{j=1}^{r} \lambda_j (g_j(x) - y) = y \left(1 - \sum_{j=1}^{r} \lambda_j \right) + \sum_{j=1}^{r} \lambda_j g_j(x).$$

The dual function:

$$q(\lambda) = \inf_{x \in X, y \in R} L(x, y, \lambda) = \begin{cases} \inf_{x \in X} \sum_{j=1}^{r} \lambda_j g_j(x) & \textit{if } \sum_{j=1}^{r} \lambda_j = 1, \\ -\infty & \textit{otherwise.} \end{cases}$$

Maximize $q(\lambda)$ subject to $\lambda \geq 0$ equivalent to:

$$\begin{aligned} \underset{\lambda}{\text{maximize}} \quad & \inf_{x \in X} \lambda' g(x) \\ \textit{subject to} \quad & \textstyle\sum_{j=1}^{r} \lambda_j = 1, \quad \lambda \geq 0. \end{aligned}$$

Let d^* be the dual optimal value. Then $v^* = d^*$. So

$$v^* \geq 0 \iff d^* \geq 0 \iff \sup_{\substack{\lambda \geq 0 \\ \sum \lambda_j = 1}} \left(\inf_{x \in X} \lambda' g(x) \right) \geq 0.$$

Since the feasible set for λ is compact, the supremum is attained by some vector μ. Thus, the condition is equivalent to the existence of $\mu \in R^r$ such that:

$$\sum_{j=1}^{r} \mu_j = 1, \quad \mu \geq 0, \quad \textit{and} \quad \inf_{x \in X} \mu' g(x) \geq 0.$$

Problem 3. (25 points)

Consider the problem

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in X, \quad g_i(x) \le 0, i = 1, \cdots, r, \end{array}$$

where X is a convex set, and f and g_js are convex over X. Assume that the problem has at least one feasible solution. Show that the following are equivalent.

(i) The dual optimal value $q^* = \sup_{\mu \in R^r} q(\mu)$ is finite.

(ii) The primal function p is proper.

(iii) The set

$$M = \left\{(u, w) \in R^{r+1} \mid \text{ there is an } x \in X \text{ such that } g(x) \le u, f(x) \le w\right\}$$

does not contain a vertical line.

Solution 3 *The primal function defined as:*

$$p(u) = \inf\{f(x) \mid x \in X, g(x) \le u\}.$$

We have $p(0) < \infty$, and $p(u)$ is not identically $+\infty$.

(ii) $\iff$ (iii): *If p is proper, then $p(u) > -\infty$ for all u. Therefore, there is no u for which $(u, w) \in M$ for all w. Thus, M does not contain a vertical line. Conversely, if M does not contain a vertical line, then for every u, the values of w such that $(u, w) \in M$ are bounded below. Thus $p(u) > -\infty$ for all u. Since p is not identically $+\infty$, p is proper.*

(ii) $\implies$ (i): *If p is proper and convex, the conjugate of a proper convex function is proper. By Weak Duality, $q(\mu) \le p(0)$. Since p is proper and a feasible solution exists, $p(0)$ is finite. Thus $\sup q(\mu) \le p(0) < \infty$.*

(i) $\implies$ (ii): *If $q^* = \sup q(\mu)$ is finite, then $q(\mu) > -\infty$ for some μ. Suppose p is not proper. Since feasible solutions exist, this means there exists $\bar{u}$ such that $p(\bar{u}) = -\infty$. Then, for any $\mu \ge 0$:*

$$q(\mu) \le \inf_x \{f(x) + \mu' g(x)\} \le p(\bar{u}) + \mu' \bar{u} = -\infty.$$

This contradicts the assumption that $q^* > -\infty$. Thus, p must be proper.

Problem 4. (25 points)

Consider a proper convex function F of two vectors $x \in R^n$ and $y \in R^m$. For a fixed $(\bar{x}, \bar{y}) \in \text{dom}(F)$, let $\partial_x F(\bar{x}, \bar{y})$ and $\partial_y F(\bar{x}, \bar{y})$ be the subdifferentials of the functions $F(\cdot, \bar{y})$ and $F(\bar{x}, \cdot)$ at $\bar{x}$ and $\bar{y}$, respectively.

(a) Show that

$$\partial F(\bar{x}, \bar{y}) \subset \partial_x F(\bar{x}, \bar{y}) \times \partial_y F(\bar{x}, \bar{y})$$

and give an example showing that the inclusion may be strict in general.

(b) Assume that F has the form

$$F(x, y) = h_1(x) + h_2(y) + h(x, y),$$

where h_1 and h_2 are proper convex functions, and h is convex, real-valued, and differentiable. Show that the formula of part (a) holds with equality.

Solution 4 ***(a)*** *Let $(u, v) \in \partial F(\bar{x}, \bar{y})$. By the definition:*

$$F(x, y) \geq F(\bar{x}, \bar{y}) + u'(x - \bar{x}) + v'(y - \bar{y}), \quad \forall x, y.$$

Set $y = \bar{y}$:

$$F(x, \bar{y}) \geq F(\bar{x}, \bar{y}) + u'(x - \bar{x}).$$

This implies $u \in \partial_x F(\bar{x}, \bar{y})$. Similarly we have $v \in \partial_y F(\bar{x}, \bar{y})$. Thus, $\partial F(\bar{x}, \bar{y}) \subset \partial_x F(\bar{x}, \bar{y}) \times \partial_y F(\bar{x}, \bar{y})$.

Example: *Consider $F(x, y) = |x + y|$ at $(\bar{x}, \bar{y}) = (0, 0)$. The subdifferential of F at $(0, 0)$ consists of vectors (u, v) such that $|x + y| \geq ux + vy$. This holds if and only if $u = v$ and $|u| \leq 1$. Thus, $\partial F(0, 0) = \{(u, u) \mid -1 \leq u \leq 1\}$. But*

$$F(x, 0) = |x| \implies \partial_x F(0, 0) = [-1, 1]$$

$$F(0, y) = |y| \implies \partial_y F(0, 0) = [-1, 1]$$

$$\implies \partial F(0, 0) \subsetneq \partial_x F(0, 0) \times \partial_y F(0, 0) = [-1, 1] \times [-1, 1].$$

(b) *Let $F(x, y) = h_1(x) + h_2(y) + h(x, y)$, we have:*

$$\partial F(x, y) = \partial(h_1(x) + h_2(y)) + \nabla h(x, y).$$

Since h_1 depends only on x and h_2 only on y:

$$\partial(h_1(x) + h_2(y)) = \partial h_1(x) \times \partial h_2(y).$$

Also, $\nabla h(x, y) = (\nabla_x h(x, y), \nabla_y h(x, y))$. Thus:

$$\partial F(\bar{x}, \bar{y}) = \{(u_1 + \nabla_x h(\bar{x}, \bar{y}), u_2 + \nabla_y h(\bar{x}, \bar{y})) \mid u_1 \in \partial h_1(\bar{x}), u_2 \in \partial h_2(\bar{y})\}.$$

And:

$$\partial_x F(\bar{x}, \bar{y}) = \partial_x(h_1(x) + h_2(\bar{y}) + h(x, \bar{y})) = \partial h_1(\bar{x}) + \nabla_x h(\bar{x}, \bar{y}),$$
$$\partial_y F(\bar{x}, \bar{y}) = \partial_y(h_1(\bar{x}) + h_2(y) + h(\bar{x}, y)) = \partial h_2(\bar{y}) + \nabla_y h(\bar{x}, \bar{y}).$$

So:

$$\partial_x F \times \partial_y F = (\partial h_1(\bar{x}) + \nabla_x h) \times (\partial h_2(\bar{y}) + \nabla_y h) = \partial F(\bar{x}, \bar{y}).$$

Problem 5. (20 points)

(Note: This exercise shows how a duality gap results in nondifferentiability of the dual function.)

Consider the problem

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in X, \quad g(x) \leq 0, \end{array}$$

and assume that for all $\mu \geq 0$, the infimum of the Lagrangian $L(x, \mu)$ over X is attained by at least one $x_\mu \in X$. Show that if there is a duality gap, then the dual function $q(\mu) = \inf_{x \in X} L(x, \mu)$ is nondifferentiable at every dual optimal solution.
Hint: If q is differentiable at a dual optimal solution μ^*, by the theory of Section 5.3 in the textbook, we must have $\partial q\left(\mu^*\right) / \partial \mu_j \leq 0$ and $\mu_j^* \partial q\left(\mu^*\right) / \partial \mu_j = 0$ for all j. Use optimality conditions for μ^*, together with any vector x_{μ^*} that minimizes $L\left(x, \mu^*\right)$ over X, to show that there is no duality gap.

Solution 5 *Assume that there is a duality gap, i.e., $q(\mu^*) < f(x^*)$, and assume that the dual function q is differentiable at a dual optimal solution μ^*.*

$$\frac{\partial q(\mu^*)}{\partial \mu_j} \leq 0 \quad \forall j, \quad \text{and} \quad \mu_j^* \frac{\partial q(\mu^*)}{\partial \mu_j} = 0 \quad \forall j.$$

By the properties of the dual function, if the infimum in the definition of $q(\mu)$ is attained at a unique point, the gradient is given by the constraint values at the minimizer of the Lagrangian. Specifically:

$$\nabla q(\mu^*) = g(x_{\mu^*}),$$

where x_{μ^*} minimizes $L(x, \mu^*)$ over X.Since $g_j(x_{\mu^*}) \leq 0$ for all j, x_{μ^*} is a primal feasible solution. Since $\mu_j^* g_j(x_{\mu^*}) = 0$ for all j, so we have complementary slackness.

$$q(\mu^*) = \inf_{x \in X} L(x, \mu^*) = L(x_{\mu^*}, \mu^*) = f(x_{\mu^*}) + \sum_{j=1}^{r} \mu_j^* g_j(x_{\mu^*}) = f(x_{\mu^*}).$$

Thus, $q(\mu^*) = f(x_{\mu^*})$, which contradicts the assumption of a duality gap. Therefore, if there is a duality gap, the dual function q cannot be differentiable at any dual optimal solution μ^*.