Homework Assignment 1

Problem 1. (25 points)

(a) Let C be a nonempty subset of $\mathbf{R}^n$, and let λ_1 and λ_2 be positive scalars. Show that if C is convex, then $(\lambda_1 + \lambda_2)\,C = \lambda_1 C + \lambda_2 C$. Show by example that this need not be true when C is not convex.
(b) Show that the intersection $\cap_{i \in I} C_i$ of a collection $\{C_i \mid i \in I\}$ of cones is a cone.
(c) Show that the image and the inverse image of a cone under a linear transformation is a cone.
(d) Show that the vector sum $C_1 + C_2$ of two cones C_1 and C_2 is a cone.
(e) Show that a subset C is a convex cone if and only if it is closed under addition and positive scalar multiplication, i.e., $C + C \subset C$, and $\gamma C \subset C$ for all $\gamma > 0$.

Answer:(a): Let $x = \lambda_1 c_1 + \lambda_2 c_2 \in \lambda_1 C + \lambda_2 C$. Consider the point $c = \frac{\lambda_1}{\lambda_1+\lambda_2} c_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} c_2$ and $\alpha = \frac{\lambda_1}{\lambda_1+\lambda_2} \in (0,1)$. Then $c = \alpha c_1 + (1-\alpha) c_2 \in C$. Thus $x = (\lambda_1 + \lambda_2)c \in (\lambda_1 + \lambda_2)C$. So $\lambda_1 C + \lambda_2 C \subset (\lambda_1 + \lambda_2)C$. Now let $x \in (\lambda_1 + \lambda_2)C$. Then $x = (\lambda_1 + \lambda_2)c$ for some $c \in C$. Consider the points $c_1 = c_2 = c$. Then $x = \lambda_1 c_1 + \lambda_2 c_2 \in \lambda_1 C + \lambda_2 C$. So $(\lambda_1 + \lambda_2)C \subset \lambda_1 C + \lambda_2 C$. Thus we have $(\lambda_1 + \lambda_2)\,C = \lambda_1 C + \lambda_2 C$.

Consider the non-convex set $C = \{0, 1\}$ in $\mathbb{R}$. Let $\lambda_1 = 1$ and $\lambda_2 = 1$. $(\lambda_1 + \lambda_2)C = (1+1)C = 2C = \{2 \cdot 0, 2 \cdot 1\} = \{0, 2\}$. $\lambda_1 C + \lambda_2 C = 1C + 1C = C + C = \{x + y \mid x \in C, y \in C\} = \{0+0, 0+1, 1+0, 1+1\} = \{0, 1, 2\}$. Since $\{0, 2\} \neq \{0, 1, 2\}$, the equality does not hold.

(b): Let $x, y \in \cap_{i \in I} C_i$ and $\alpha, \beta \geq 0$. Then $x, y \in C_i$ for all $i \in I$. Since each C_i is a cone, $\alpha x + \beta y \in C_i$ for all $i \in I$. Thus $\alpha x + \beta y \in \cap_{i \in I} C_i$. So $\cap_{i \in I} C_i$ is a cone.

(c): Let C be a cone in $\mathbb{R}^n$ and A be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Let $x, y \in A(C)$ and $\alpha, \beta \geq 0$. Then there exist $u, v \in C$ such that $x = A(u)$ and $y = A(v)$. Since C is a cone, $\alpha u + \beta v \in C$. Thus $\alpha x + \beta y = \alpha A(u) + \beta A(v) = A(\alpha u + \beta v) \in A(C)$. So $A(C)$ is a cone.

(d): Let $x, y \in C_1 + C_2$ and $\alpha, \beta \geq 0$. Then there exist $u_1, v_1 \in C_1$ and $u_2, v_2 \in C_2$ such that $x = u_1 + u_2$ and $y = v_1 + v_2$. Since C_1 and C_2 are cones, $\alpha u_1 + \beta v_1 \in C_1$ and $\alpha u_2 + \beta v_2 \in C_2$. Thus $\alpha x + \beta y = \alpha(u_1 + u_2) + \beta(v_1 + v_2) = (\alpha u_1 + \beta v_1) + (\alpha u_2 + \beta v_2) \in C_1 + C_2$. So $C_1 + C_2$ is a cone.

(e): Suppose C is a convex cone. Then for any $x, y \in C$ and $\alpha, \beta \geq 0$, we have $\alpha x + \beta y \in C$. Setting $\alpha = \beta = 1$, we get $x + y \in C$. Thus $C + C \subseteq C$. Setting $y = 0$ and $\beta = 0$, we get $\alpha x \in C$ for all $\alpha > 0$. Thus $\gamma C \subseteq C$ for all $\gamma > 0$.

Problem 2. (15 points)

Let C be a nonempty convex subset of $\mathbf{R}^n$. Let also $f = (f_1, \ldots, f_m)$, where $f_i : C \mapsto \mathbf{R}$, $i = 1, \ldots, m$, are convex functions, and let $g : \mathbf{R}^m \mapsto \mathbf{R}$ be a function that is convex and monotonically nondecreasing over a convex set that contains the set $\{f(x) \mid x \in C\}$, in the sense that for all u_1, u_2 in this set such that $u_1 \leq u_2$, we have $g(u_1) \leq g(u_2)$. Show that the function h defined by $h(x) = g(f(x))$ is convex over C. If in addition, $m = 1, g$ is monotonically increasing and f is strictly convex, then h is strictly convex.

Answer: For any $x, y \in C$ and $\alpha \in [0, 1]$, $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$. So $g(f(\alpha x + (1 - \alpha)y)) \leq g(\alpha f(x) + (1 - \alpha)f(y))$. Using the fact that g itself is a convex function, we have $g(\alpha f(x) + (1 - \alpha)f(y)) \leq \alpha g(f(x)) + (1 - \alpha)g(f(y))$. Chaining these inequalities together confirms that h is convex.

To the second case, the strict convexity of f makes the first inequality strict, $f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y)$. Since g is monotonically increasing, this strictness is preserved, leading to $g(f(\alpha x + (1 - \alpha)y)) < g(\alpha f(x) + (1 - \alpha)f(y))$. This, combined with the convexity of g, results in the overall strict inequality $h(\alpha x + (1 - \alpha)y) < \alpha h(x) + (1 - \alpha)h(y)$, thus proving that h is strictly convex.

Problem 3. (20 points)

(a) Consider the quadratic program

$$\begin{array}{ll} \underset{x}{\text{minimize}} & 1/2|x|^2 + c'x \\ \text{subject to} & Ax = 0 \end{array}$$

where $c \in \mathbf{R}^n$ and A is an $m \times n$ matrix of rank m. Use the Projection Theorem to show that

$$x^* = -\left(I - A'\left(AA'\right)^{-1} A\right) c$$

is the unique solution.

(b) Consider the more general quadratic program

$$\begin{array}{ll} \underset{x}{\text{minimize}} & 1/2(x - \bar{x})'Q(x - \bar{x}) + c'(x - \bar{x}) \\ \text{subject to} & Ax = b \end{array}$$

where c and A are as before, Q is a symmetric positive definite matrix, $b \in \mathbf{R}^m$, and $\bar{x}$ is a vector in $\mathbf{R}^n$, which is feasible, i.e., satisfies $A\bar{x} = b$. Use the transformation $y = Q^{1/2}(x - \bar{x})$ to write this problem in the form of part (a) and show that the optimal solution is

$$x^* = \bar{x} - Q^{-1}\left(c - A'\lambda\right)$$

where λ is given by

$$\lambda = \left(AQ^{-1}A'\right)^{-1} AQ^{-1}c$$

(c) Apply the result of part (b) to the program

$$\begin{array}{ll} \underset{x}{\text{minimize}} & 1/2x'Qx + c'x) \\ \text{subject to} & Ax = b \end{array}$$

and show that the optimal solution is

$$x^* = -Q^{-1}\left(c - A'\lambda - A'\left(AQ^{-1}A'\right)^{-1} b\right)$$

Answer: (a): $\frac{1}{2}|x|^2 + c'x$ is equivalent to the distance $|x - (-c)|$. The problem can therefore be interpreted as finding the point in the subspace defined by $Ax = 0$ that is closest to the point $-c$. By the Projection Theorem, the unique solution x^* is the orthogonal projection of $-c$ onto this subspace $N(A)$. Since the projection matrix onto the range space of A' is $A'(AA')^{-1}A$, the projection of $-c$ onto $N(A)$ is $x^* = (I - A'(AA')^{-1}A)(-c)$.

(b): Use transformation $y = Q^{1/2}(x - \bar{x})$. The objective function's quadratic term transforms into $\frac{1}{2}|y|^2$, and the linear term becomes $(Q^{-1/2}c)'y$. This transformed problem has the exact form as the problem in part (a), allowing us to directly apply its solution to find the optimal y^*. Transforming back via $x^* - \bar{x} = Q^{-1/2}y^*$ we have $x^* - \bar{x} = -Q^{-1}c + Q^{-1}A'(AQ^{-1}A')^{-1}AQ^{-1}c$. By using the given definition for λ as $\lambda = (AQ^{-1}A')^{-1}AQ^{-1}c$, the expression simplifies to the final optimal solution $x^* = \bar{x} - Q^{-1}(c - A'\lambda)$.

(c): To apply the result of part (b) to the program that minimizes $\frac{1}{2}x'Qx + c'x$ subject to $Ax = b$, we must align its objective function with the form from part (b). The objective in part (b) can be expanded, ignoring constants, to be equivalent to minimizing $\frac{1}{2}x'Qx + (c_b - Q\bar{x})'x$. To match the objective of the current problem, we must set $c_b - Q\bar{x} = c$, which implies $c_b = c + Q\bar{x}$ for some feasible $\bar{x}$. We now use the solution formula from part (b), $x^* = \bar{x} - Q^{-1}(c_b - A'\lambda_b)$, where $\lambda_b = (AQ^{-1}A')^{-1}AQ^{-1}c_b$. Using the feasibility condition $A\bar{x} = b$, we find that $\lambda_b = (AQ^{-1}A')^{-1}(AQ^{-1}c + b)$.

Problem 4. (20 points)

(a) Let C be a nonempty convex cone. Show that $\mathrm{cl}(C)$ and $\mathrm{ri}(C)$ is also a convex cone.
(b) Let $C = \mathrm{cone}(\{x_1, \ldots, x_m\})$. Show that

$$\mathrm{ri}(C) = \left\{ \sum_{i=1}^{m} a_i x_i \mid a_i > 0, i = 1, \ldots, m \right\}.$$

Answer: (a): For $\mathrm{cl}(C)$, the closure of a convex set is convex. To show it is a cone, let $x \in \mathrm{cl}(C)$ and $\lambda > 0$. There exists a sequence $\{x_k\}$ in C converging to x. Since C is a cone, the sequence $\{\lambda x_k\}$ also lies in C, and it converges to λx, which implies $\lambda x \in \mathrm{cl}(C)$. For $\mathrm{ri}(C)$, the relative interior of a convex set is also convex. To show it is a cone, let $x \in \mathrm{ri}(C)$ and $\lambda > 0$. This means there is an open ball around x, intersected with the affine hull $\mathrm{aff}(C)$, that is contained within C. By scaling this ball and its center by the factor λ, we can construct a new open ball centered at λx. This new ball is also contained in C. This implies that $\lambda x \in \mathrm{ri}(C)$, confirming that $\mathrm{ri}(C)$ is a convex cone.

(b): To show that for $C = \mathrm{cone}(\{x_1, ..., x_m\})$, the relative interior is $ri(C) = \{\sum_{i=1}^{m} a_i x_i | a_i > 0, i = 1, ..., m\}$, we can interpret the cone C as the image of the non-negative orthant in $\mathbb{R}^m$, denoted $\mathbb{R}^m_+$, under the linear transformation A whose columns are the vectors $x_1, ..., x_m$. A standard result from convex analysis states that for a linear map A and a convex set S, the relative interior of the image is the image of the relative interior, i.e., $ri(A(S)) = A(ri(S))$. In this context, $C = A(\mathbb{R}^m_+)$, and the relative interior of the non-negative orthant, $ri(\mathbb{R}^m_+)$, is the set of all vectors with strictly positive components. Applying the theorem, $ri(C) = A(ri(\mathbb{R}^m_+))$, which is precisely the set of all linear combinations of the vectors $x_1, ..., x_m$ with strictly positive coefficients.

Problem 5. (10 points)

Let X be a nonempty bounded subset of $\mathbf{R}^n$. Show that

$$\mathrm{cl}(\mathrm{conv}(X)) = \mathrm{conv}(\mathrm{cl}(X)).$$

In particular, if X is compact, then $\mathrm{conv}(X)$ is compact.

Answer: To show $\mathrm{conv}(\mathrm{cl}(X)) \subseteq \mathrm{cl}(\mathrm{conv}(X))$, consider $y = \sum \alpha_i y_i \in \mathrm{conv}(\mathrm{cl}(X))$ where points $y_i \in \mathrm{cl}(X)$. For each y_i, there exists a sequence $\{x_{i,j}\}$ in X that converges to y_i. The sequence of points $z_j = \sum \alpha_i x_{i,j}$ lies entirely in $\mathrm{conv}(X)$, and this sequence converges to y. Therefore, y must be in the closure, $\mathrm{cl}(\mathrm{conv}(X))$.

To show $\mathrm{cl}(\mathrm{conv}(X)) \subseteq \mathrm{conv}(\mathrm{cl}(X))$, we note that since X is bounded, its closure $\mathrm{cl}(X)$ is compact. The convex hull of a compact set is also compact, which implies that $\mathrm{conv}(\mathrm{cl}(X))$ is a closed set. Since $X \subseteq \mathrm{cl}(X)$, it follows that $\mathrm{conv}(X) \subseteq \mathrm{conv}(\mathrm{cl}(X))$. Because $\mathrm{conv}(\mathrm{cl}(X))$ is a closed set containing $\mathrm{conv}(X)$, it must also contain its closure. The equality is thus established.

If X is compact, it is closed and bounded. Since X is closed, $\mathrm{cl}(X) = X$, so the equality gives $\mathrm{cl}(\mathrm{conv}(X)) = \mathrm{conv}(X)$, meaning $\mathrm{conv}(X)$ is closed. The convex hull of a bounded set is bounded, so $\mathrm{conv}(X)$ is both closed and bounded, and therefore compact.

Problem 6. (10 points)

Let C_1 and C_2 be convex sets. Show that

$$C_1 \cap \mathrm{ri}(C_2) \neq \emptyset \text{ if and only if } \mathrm{ri}(C_1 \cap \mathrm{aff}(C_2)) \cap \mathrm{ri}(C_2) \neq \emptyset.$$

Answer: "$\Longleftarrow$": if a point x exists in $\mathrm{ri}(C_1 \cap \mathrm{aff}(C_2)) \cap \mathrm{ri}(C_2)$, then by definition x is in $\mathrm{ri}(C_2)$ and also in $C_1 \cap \mathrm{aff}(C_2)$, which implies $x \in C_1$. Therefore, x belongs to $C_1 \cap \mathrm{ri}(C_2)$, so the intersection is nonempty.

"$\Longrightarrow$": assume there exists a point $x \in C_1 \cap \mathrm{ri}(C_2)$. This means $x \in C_1$ and $x \in \mathrm{ri}(C_2)$. Since any point in the relative interior of a set must also be in its affine hull, $x \in \mathrm{aff}(C_2)$. Consequently, x belongs to the set $C = C_1 \cap \mathrm{aff}(C_2)$. We now have a nonempty intersection between the convex set C and the relative interior of the convex set C_2. Since intersection $C \cap \mathrm{ri}(C_2)$ is nonempty, it follows that the intersection of their relative interiors $\mathrm{ri}(C) \cap \mathrm{ri}(C_2)$ which is equal to $\mathrm{ri}(C_1 \cap \mathrm{aff}(C_2)) \cap \mathrm{ri}(C_2)$, must also be nonempty.